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NEGATIVE EXTERNALITIES, COMPETITION, AND CONSUMER CHOICE*

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Consumers sometimes make choices that impose greater external costs on those who do not make the same choice. This paper examines how the *selectivity* of negative externalities in such situations affects the competitive equilibrium and the desirability of an externality-reducing public policy. Selective negative externalities create network externalities, but outcomes may differ greatly from typical network effects. Price effects may cause the imposing product's sales to decline with the size of the negative externality. Consequently, a positive competitive effect may overwhelm the externality's negative direct effects on welfare, such that a policy that enlarges the externality may improve welfare.

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I. INTRODUCTION

THIS PAPER RELATES NEGATIVE EXTERNALITIES AND COMPETITION. In a competitive environment with differentiated products, the tendency of consumers to impose costs on others and not account for these costs in their decisions may have implications not just for the allocative efficiency of market outcomes, but also, in meaningful ways, for the structure of the equilibrium. The competitive effects of externalities alter firms' incentives with respect to prices, outputs, and the imposition of external cost itself. Consequently, as this paper demonstrates, competitive effects alter and may even reverse the marginal effects of negative externalities on social welfare. Accounting for these effects forces a re-examination of the public policy solutions that address problems of shared resources in a wide range of markets.

In many market situations, consumers make choices that impose greater external costs on those who do not make the same choice that they do, as compared with those who do. Consider first what one might call "combatant goods": products that bundle greater imposition of external costs with greater protection against the same costs, relative to alternatives. Sport utility vehicles (SUVs) impose greater risks of injury and death on other motorists than do cars, while at the same time providing their occupants with increased protection against these risks relative to cars. Visible car-theft deterrent devices (such as the Club) tend to push thieves to other cars, including those protected by invisible deterrent devices (such as Lojack); thus they redistribute crime rather than reducing it, while inoculating their owners against others who pursue the same strategy (see Ayres and Levitt [1998]).

Another instance of the phenomenon is presented by situations in which a consumer's adoption of a noisome product reduces her displeasure from others' use of the product, either because personal use increases habituation to others' use through a physiological mechanism, such as addictive tolerance; or because it changes attitudes through a psychological mechanism, such as reaction to cognitive dissonance.[1] For example, Beh [1989] observes that smokers report finding secondhand smoke less objectionable than do non-smokers, suggesting a cause-and-effect relationship between personal tobacco use and less-aggrieved attitudes towards others' use. Meanwhile, people who maintain their lawns with gasoline-powered mowers and leaf blowers might, because they have consequently adjusted their attitudes, be less likely to find the fumes and noise from neighbors' equipment objectionable. Similarly, the fact that one's neighbors pollute a shared lake through pesticide runoff or by using motorboats may be viewed less unfavorably if one is doing these things oneself. In such situations, as with combatant goods, the noisome product imposes greater costs on individuals who have not adopted it as compared with those who have.

Yet another instance can be found in situations in which agents who do not adopt a product or service incur a stigma that grows stronger, perhaps by accruing credibility or acceptance, as additional agents adopt. As businesses obtain certifications (e.g., ISO 9000, web-based certifications, etc.), their uncertified competitors increasingly run the risk of being viewed as unsafe or undesirable to deal with.[2] As the job market in a certain profession becomes engorged with candidates who have degrees from prestigious institutions, applicants that do not have such degrees are increasingly viewed as deficient. Analogous effects occur, generally, with respect to *all* goods to which social norms are attached, to the extent that incremental adoption increases sanctioning against non-adopters. Thus, as more people wear expensive suits to interviews, or give their co-workers a bottle of wine at the holidays, such practices increasingly become the norm, and those who do not engage in them face increased social costs relative to those who do.[3]

While many of these types of situations have been discussed by economists, they have not previously been recognized as involving anything other than simple negative externalities.[4] But, because the external costs are imposed *selectively* (i.e., more so on those not making the same choice), they exhibit competitive effects not identified with non-selective negative externalities. Specifically, selective negative externalities imply network externalities: an increase in the number of people choosing an activity that imposes a selective external cost increases the benefit of choosing the activity relative to the best alternative.[5] The result is what one might call an "if-you-can't-beat-'em-join-'em" (IYCBEJE) network effect, whereby consumers join a bandwagon created by the increased undesirability of the alternative choice. Intuition suggests that the seller has an incentive to enlarge external costs (or consumers' perceptions of them) if the costs are selective, because doing so would increase his sales.[6]

The paper uses a simple location model of differentiated products to examine the effect of selective, consumer choice-imposed negative externalities on the competitive market equilibrium and the strategic behaviors of the competitors. The framework allows us to consider social welfare outcomes and the effect of a prospective public policy that reduces the externality at the margin. Contrary to the IYCBEJE intuition, one finds that, depending upon consumers' relative demand for the competing products, the price effects of selective negative externalities may cause the imposing product's sales to decline, rather than rise, with the size of the externality. Consequently, positive competitive effects may overwhelm the externalities' negative direct effects on welfare, such that an externality-*enlarging* policy actually improves welfare.

Selective negative externalities and their resulting competitive effects are appropriately viewed through the lens of compatibility. When a consumer considers whether to use a selective-externality-imposing product or some alternative, she typically considers how well or badly the alternative will fare in terms of its relationship to the imposing product and how many units of the imposing product there are in use. For example, the prospective buyer of a car might wonder how well she will make out if she collides with an SUV, and how many SUVs she is likely to encounter on the road. The first question has to do with compatibility, and the second with the size of the relevant installed base.[7]

Past analyses of network externalities have generally focused on user-specific positive externalities of consumption, that is, situations in which users of a good benefit other people who use the same good or a compatible good. Several papers in this vein analyze consumers' equilibrium market choices, positing the importance of compatibility benefits as a parameter of interest (e.g., Katz and Shapiro [1985, 1992]; Farrell and Saloner [1992]; Crémer, Rey and Tirole [2000]). Compatibility between different products is then assumed to be a choice variable of the consumer (through the decision to buy a converter) or the firm (through the decision to build a product bundled with a converter, or else to agree jointly with other firms on a product standard), whence these papers examine the compatibility decision as part of a set of equilibrium outcomes.

While likewise analyzing consumers' choices in equilibrium, the present paper takes *incompatibility* between different products as given. It breaks the importance of incompatibility costs into two components: the size of the negative externality, which it takes as a choice variable of the firm; and the level of selectivity of the externality, which it treats as an exogenous parameter of interest. The size of the externality, which the other papers treat as wholly exogenous, is thus an outcome of our equilibrium analysis. There are two reasons for this modeling approach. First, it is consistent with the conventional analysis of negative externalities, which treats the size of the externality as a variable that may be manipulated by private parties. In this context, public policy options for providing incentives to mitigate the externality may be considered, using marginal analysis of the social costs and benefits. Second, with respect to the various examples of selective negative externalities proposed earlier, it is realistic to think of the negative externality as being a variable that firms may practically manipulate.[8]

Another important distinction between the present paper and some of the compatibility literature pertains to the assumed effects of compatibility on prices. Katz and Shapiro [1985], Crémer, Rey and Tirole [2000], and Hogendorn and Yuen [2009], among others, assume that firms that compete in outputs "*à la Cournot*," such that all firms have positive sales only if their quality-adjusted prices are all equal. This vacates the possibility of the compatibility decision having any price effects. Cournot competition is a good assumption where products are perceived to be homogenous but for network size expectations (for example, Crémer, Rey and Tirole [2000] model competition between Internet backbone providers). However, it is less-good in most of the consumer product contexts where selective negative externalities are likely to be relevant (for example, the markets for motor vehicles and outdoor power equipment). In such contexts, products are typically differentiated, and consumers' tastes are heterogeneous. Accordingly, as discussed above, the present paper uses a location model *à la Hotelling.* It is shown, in that context, that the imposing firm's chosen level of incompatibility cost (that is, the size of the negative externality) affects quality-adjusted prices in a way that depends upon consumers' relative demand for one product over another – whence our result that the effect of the negative externality on sales and welfare, at the margin in equilibrium, depends on relative demand.

The rest of the paper is organized as follows. Section II describes the basic model. Section III analyzes the model's equilibrium and discuss its public policy implications. Conclusions are presented in section IV.

II. A MODEL OF SELECTIVE NEGATIVE EXTERNALITIES

II (i). *Brief Summary of the Model*

Selective negative externalities are modeled in a differentiated products framework in which consumers exhibit heterogeneous preferences based on their "location," as represented on a unit interval. Two products are offered. One imposes an external cost that reduces the utility of all consumers, but its effects are selective: buyers of the product incur a smaller cost than buyers of the rival product. The other product imposes no external cost. The manufacturer of the imposing product may choose how large to make the external cost, given the degree of its selectivity. Relative demand for the imposing product is posited as an exogenous parameter.

Firm decision-making occurs in two stages. First, the manufacturer of the imposing product sets the level of the external cost. Second, prices are set taking the external cost as given. This assumption is justified based on a notion that external costs are set through long-term design or marketing decisions, whereas prices are set in a spot game between players in the market.

Two cases are considered: one where the imposing product competes with a free product (or no product), and one where the imposing product competes on price with a product supplied by another firm (i.e., a "market product"). The first case might represent such scenarios described in the introduction as the decision to smoke (versus not), the decision to apply pesticides to one's land (versus not), and the decision to get an ISO 9000 certification (versus not). The second case represents scenarios such as the decision to purchase a visible car-theft deterrent device versus an invisible device; the decision to purchase a gasoline-powered mower versus an electric mower; and the decision to wear an expensive suit to an interview, marketed by a high-end fashion designer, versus a more modest suit, marketed under a department store label.[9]

When the imposing product competes with a free product, the negative externality has no price effect. Consequently, an increase in the size of the negative externality increases the number of consumers choosing the imposing product. This is consistent with the ICYBEJE intuition that selective negative externalities create a bandwagon effect favoring the imposer. And, as in the standard case of non-selective external costs, the negative externality is set too large by private parties, thus a policy that reduces it improves welfare.

However, when the imposing product competes on price with another market product, the negative externality enlarges the price differential between the two products because it effectively degrades the rival product in relative terms. It follows that a larger negative externality draws consumers to the imposing product if it is in high relative demand, but drives consumers away if demand for the imposing product is low. This is because, while the desire to avoid incompatibility vis-à-vis the negative externality is paramount for consumers when the installed base of the imposing product is large, it becomes less important when the installed base is small, whence the externality's price effect takes precedence. As a consequence, contrary to normal expectations about unregulated market equilibrium, a policy to *enlarge* external costs improves welfare when demand for the imposing product is low enough. What is happening is that an enlargement in the size of the negative externality reduces the number of units of imposing product in use to a sufficient extent that welfare improves, even though each unit of the product is imposing a larger negative externality.

II. (ii). *Assumptions*

Consider a market for two products, A and B, sold at prices p_A and p_B, respectively. Consumers are distributed uniformly on the interval (0,1) based on their preferences for A versus B, with the total number of consumers normalized to 1. Consumers choose whether to purchase A or B; each consumer will choose at most one unit of one of the two products (i.e., there is no outside good). It is assumed that product A, when used, imposes costs on other consumers. Specifically, consumers incur costs proportional to the total number of consumers who purchase A.[10] However, purchasing A shields the consumer against a fixed portion of the costs imposed by other units of A. Consumers always bear their own costs and are never liable for costs imposed on others.

A consumer located at a point j ($1 \geq j \geq 0$) who purchases A receives utility

(1) $$U_A(j) = v + \theta - t(1-j) - (1-\sigma)\lambda Q_A - p_A$$

and if she purchases B she receives instead

(2) $$U_B(j) = v - \theta - tj - \lambda Q_A - p_B$$

Here, Q_A is the number of consumers who purchase A; λ is the cost that an individual unit of A imposes on other consumers ($\lambda \geq 0$); σ represents the shielding that product A provides its purchasers ($1 \geq \sigma \geq 0$); v represents the demand for all products; θ, which may be positive or negative, parameterizes the exogenous demand for A relative to B; and t represents the intensity of consumers' relative preferences for A or B ($t > 0$). A consumer who chooses neither A nor B receives utility of zero. Each consumer makes the choice that maximizes her utility.

Product A is produced by a single firm, firm A. For simplicity, the marginal cost of production for A is assumed constant at zero.[11] However, firm A incurs a fixed cost of production, $C(\lambda)$, which depends upon the negative externality imposed by A.[12] The fixed cost technology is specified by $C'(\lambda) = k(\lambda - \underline{\lambda})$, where $k, \underline{\lambda} > 0$. This implies $C(.)$ is convex everywhere, increasing on $\lambda > \underline{\lambda}$, and decreasing on $\lambda < \underline{\lambda}$. The chosen function has the intuitively-appealing properties that it is (symmetrically) costly for firm A to set λ greater than or less than $\underline{\lambda}$, and increasingly costly the farther λ is set from this minimum-cost level.[13] To eliminate unnecessary complexity in the solution of the model, "no-tipping" conditions are imposed, limiting attention to σ sufficiently small that firm A in most cases has no incentive to use λ to tip the market in response to a small change in exogenous parameters (see Appendix for derivations).

Firm A first sets λ to maximize

(3) $$\Pi_A = p_A Q_A - C(\lambda)$$

then, it sets p_A to maximize (3), taking its previous choice of λ as given.

III. EQUILIBRIUM

III (i). *Externality-Imposing Market Product versus Free Product*

Let us begin with the case in which product B is supplied for free, that is, $p_B = 0$. B may be a product provided and subsidized by the government, just something that is freely available, or it might represent the possibility of obtaining no product as an alternative to product A.

Let us assume v is large enough that all consumers choose to purchase A or B at equilibrium prices, that is, $Q_A + Q_B = 1$, where Q_B is the number of consumers that purchase B.[14] Combining (1) and (2) then reveals

(4) $$\Psi_j^{MF} = 2\theta + \sigma\lambda Q_A - t(1-2j)$$

as the consumer's reservation price for A relative to B. Note that, if $\sigma\lambda > 0$, then Ψ_j^{MF} increases with Q_A. Thus a network externality may be said to exist for A if $\sigma\lambda > 0$.[15] The network externality is the product of two components: a negative externality (λ) due to A; and a selectivity parameter (σ) that represents the extent to which the negative externality affects only the non-purchasers of A (i.e., purchasers of B).

Following the standard Hotelling model analysis, we may use (4) to derive A's demand equation[16]

(5) $$Q_A = \frac{t - p_A + 2\theta}{2t - \sigma\lambda}$$

We proceed by substituting (5) into (3) and taking the first-order condition corresponding to firm A's second-stage profit maximization (i.e., taking λ as given). This yields $p_A^* = \theta + \frac{t}{2}$. Substituting back into A's profit function, we then obtain A's first-stage first-order condition corresponding to its choice of λ:

(6) $$p_A^* \frac{\sigma Q_A^*}{2t - \sigma\lambda^*} = C'(\lambda^*)$$

Since (6) does not have a closed-form solution, we must characterize the equilibrium level of the negative externality, λ^*, by analytical methods (see Appendix). Figure 1 summarizes the resulting equilibrium given values of the parameters θ and σ, subject to the no-tipping condition imposed on σ.

(Place Figure 1 approximately here.)

Our focus is on the effect of small changes in the equilibrium level of the negative externality on A's price and sales and, related to these, on welfare, in the region in Figure 1 in which the market is split between A and B. Because λ is assumed fixed at the second stage, we may analyze these effects as if λ were an exogenous variable, much as one does when performing comparative static analysis. As p_A is obviously invariant to λ, we proceed to the effect of λ on A's sales, obtained simply by substituting $p_A^* = \theta + \frac{t}{2}$ into (5) and differentiating with respect to λ. We find:

Proposition 1 (*Competitive Effect of Selective Externality ['CESE'], Imposing Market Product vs. Free Product*). When the negative externality is selective, an increase in its size is associated with greater sales for the imposing product.

The welfare implications of the privately-chosen level of the negative externality come from differentiating the following standard social welfare function with respect to λ:

(7) $$W \equiv \Pi_A + \int_{j^*}^{1} U_A(j)\,dj + \int_{0}^{j^*} U_B(j)\,dj$$

We find that[17]

Proposition 2 (Welfare Result, Imposing Market Product vs. Free Product). The producer of the imposing product sets the negative externality too high. A public policy to reduce it improves welfare.

Neither result is surprising. Indeed, Proposition 2 is consistent with the conventional, competition-agnostic analysis of externalities. Moreover, both results hold for more general specifications of the utility and cost functions. Suppose we replace $-(1-\sigma)\lambda Q_A$ in (1) and

$-\lambda Q_A$ in (2) with general functions $f^A(\lambda,Q_A)$ and $f^B(\lambda,Q_A)$, respectively. Assume each function is decreasing in both λ and Q_A with negative cross-partials on the relevant domain, and that each takes a value of zero if either argument is zero. To incorporate selectivity, assume that the respective partial derivatives and cross-partials are strictly greater for $f^A(\lambda,Q_A)$ than $f^B(\lambda,Q_A)$, except of course at $\lambda=0$ or $Q_A=0$. Then both propositions follow, so long as $h(\lambda,Q_A)\equiv f^A-f^B$ is concave, linear, or not too convex; there is no need to impose any restrictions with respect to the cost function, C.[18]

III (ii). *Externality-Imposing Market Product versus Non-Imposing Market Product*

We now consider equilibrium in the case in which product B is supplied by a competing firm, firm B. Firm B's marginal cost of production in zero. It sets p_B to maximize

(8)
$$\Pi_B = p_B Q_B$$

Again assume v is large enough that all consumers will choose to purchase A or B at equilibrium prices, implying $Q_A+Q_B=1$.[19] Combining (1) and (2) reveals

(9)
$$\Psi_j^{MM} = 2\theta+\sigma\lambda Q_A - t(1-2j)+p_B$$

as the consumer's relative reservation price for A, and again a network externality exists for A if $\sigma\lambda>0$.

Use (9) to derive demand equations for both A *and* B, to wit,

(10)
$$Q_A=\frac{t-p_A+p_B+2\theta}{2t-\sigma\lambda};\ Q_B=\frac{t-\sigma\lambda+p_A-p_B-2\theta}{2t-\sigma\lambda}$$

and make substitutions into the corresponding profit functions. The firms' first-order conditions with respect to price, taking λ as given, together yield

(11) $$p_A^* = t + \tfrac{2}{3}\theta - \tfrac{1}{3}\sigma\lambda; \;\; p_B^* = t - \tfrac{2}{3}\theta - \tfrac{2}{3}\sigma\lambda$$

Substituting back into A's profit function, we obtain A's first-order condition with respect to λ:

(12) $$p_A^* \frac{\sigma\left(Q_A - \frac{2}{3}\right)}{2t - \sigma\lambda^*} = C'\left(\lambda^*\right)$$

Note that (12) shows that A will only set $\lambda > \underline{\lambda}$ when $Q_A > \frac{2}{3}$ under the resulting equilibrium. This suggests a relationship between size of the installed base and incentives to set a large negative externality in equilibrium; this relationship will be demonstrated more formally below.

As in the market product versus free product (hereafter "free-B") case, it is necessary to characterize the equilibrium λ^* by analytical methods (see Appendix). Figure 2 displays the resulting equilibrium given values of the parameters θ and σ, subject to the no-tipping condition imposed on σ.

(Place Figure 2 approximately here.)

The focus of our analysis now is on the effect of small changes in the equilibrium level of the negative externality on the relative prices of the two products, the sales of A, and, related to these, on welfare; results will pertain to the region in Figure 2 in which the market is split between A and B.

Using (11) and differentiating with respect to λ, we first obtain

Proposition 3. The price differential between the imposing product and the rival product increases with the size of the negative externality.

To understand why this happens, let us derive the inverse demand curves for A and B by rearranging the respective demand equations in (10),

(13) $$p_A = t + p_B + 2\theta - \left(2t - \sigma\lambda\right)Q_A$$

(14) $$p_B = t + p_A - 2\theta - \sigma\lambda - \left(2t - \sigma\lambda\right)Q_B$$

Two effects may be observed from an increase in λ. First, it causes an outward rotation of both A's and B's inverse demand curves and, therefore, *intensifies price competition*. Because the model assumes constant total market sales, this has no significance to the equilibrium other than implying a net transfer from firms to consumers. Second, an increase in λ causes an inward *parallel* shift in B's inverse demand curve. This reflects the notion that the network externality "degrades" product B relative to product A. Considering the utility functions (1) and (2) and recalling $Q_A = 1 - Q_B$, we see that an increase in λ reduces the consumer's relative preference for B proportional to σ. It follows that an increased negative externality causes an unambiguous reduction in the price of B. This, in turn, implies an inward parallel shift in *A's* inverse demand curve. This means A's price drops as well, though by less than the amount of the reduction in B's price, hence the increased price differential.

The implications of the price effect of the negative externality for A's sales are articulated in the following formal results:[20]

Proposition 4 (Competitive Effect of Selective Externality ["CESE"], Imposing Market Product vs. Market Product). When the negative externality is selective, an increase in its size is associated with greater sales for the imposing product when demand for that product is high (i.e., $\theta > -\frac{t}{2}$), but lower sales when demand for the imposing product is low (i.e., $\theta < -\frac{t}{2}$).

Corollary 4.1. An increase in the selective negative externality has a positive direct effect on the sales of the imposing product, holding prices constant; and a negative indirect effect through the price differential between the products. The direct effect is larger than the indirect effect when $\theta > -\frac{t}{2}$, but smaller when $\theta < -\frac{t}{2}$.

Welfare implications of the privately-chosen level of λ in the current case are derived using the appropriately-defined welfare function,

$$W \equiv \Pi_A + \Pi_B + \int_{j^*}^{1} U_A(j)\,dj + \int_{0}^{j^*} U_B(j)\,dj \tag{15}$$

which is identically equal to (7), as, relative to the free-B case, $p_B Q_B$ is simply transferred from the total surplus of B's consumers to firm B's profit. We find that[21]

Proposition 5 (Welfare Result, Imposing Market Product vs. Market Product). There exists a threshold $\underline{\theta}\left(\sigma,\underline{\lambda},t,k\right) < -\frac{t}{2}$ such that, if demand for the imposing product is lower than $\underline{\theta}$, its producer sets the negative externality too low. In this case, a public policy to enlarge the externality improves welfare.

To understand these results better, let us consider a numerical example, positing values for the parameters. Let $v = 7$, $t = 1$, $k = 1$, $\underline{\lambda} = 1$, and $\sigma = \frac{1}{2}$. This makes the critical value for θ in the Proposition 4, $\theta = -\frac{t}{2} = -\frac{1}{2}$; and the value for θ corresponding to the edge of the $Q_A = 0$ corner solution, $\theta = \frac{\sigma\underline{\lambda} - 3t}{2} = -1.25$.[22] Solving the posited parameter values into the price and quantity equations (10) and (11), and then into the first-order condition for λ (12), one may directly obtain the equilibrium values of the negative externality λ^* corresponding to different levels of imposing product demand θ. One may then observe the effects of a small exogenous increase in λ on the price differential between the products, the sales of the imposing product, and welfare at each value of θ.

Table I displays the equilibrium values λ^* and effects of a 0.01 increase in λ for six posited values for θ, including the critical value $\theta = -\frac{1}{2}$. The sign of the change in $p_A - p_B$, Q_A, and W is shown in each case.

(Place Table I approximately here.)

We observe that an increase in the negative externality always increases the price differential between the imposing product and its rival. Meanwhile, the effect of the externality on sales of the imposing product depends upon that product's relative demand. Additional light may be shed on the latter finding by breaking $\frac{\partial Q_A}{\partial \lambda}$ into the direct and indirect effect components described in Corollary 4.1 and by considering how each component varies with the different values of θ posited in our numerical example. In the Appendix it is shown that the direct effect component equals $\frac{\sigma Q_A}{2t - \sigma\lambda}$, while the indirect effect is $-\frac{\sigma}{3(2t - \sigma\lambda)}$. The values of these, using our posited values for the parameters, are shown in the last two columns of Table I. The indirect effect through the price differential is always negative as expected; it is not monotonic in θ (it is minimized at $\theta = -\frac{1}{2}$), but it varies little with θ. Meanwhile, the positive direct effect varies substantially with θ, and its relationship to θ is monotone positive: intuitively, the direct effect is the network effect of the imposing product, which is based on its installed base. Thus, the example suggests that the size of the imposing product's installed base, which varies positively with demand, is the main determinant of the sign of the effect of the negative externality on the imposing product's sales.

The numerical example also sheds light on the welfare result stated in Proposition 5. Consider a decomposition of the welfare effect of a change in the negative externality as the sum of an indirect effect through the sales of the imposing product A and a direct effect holding A's sales constant. The direct effect of the negative externality on welfare is always negative, as a larger negative externality imposed per unit of product A unequivocally makes consumers worse off.

But the indirect effect may be positive or negative: people will be made worse off if the externality increases the sales of A, but better off if it decreases A's sales. The effects of the negative externality on A's sales at different levels (θ) of demand for A, shown in Table I, indicate the sign of the indirect component and provide intuition on why the overall welfare effects vary with demand. When demand for A is high, as at $\theta = 0.75$ in our example, an increase in the size of the negative externality increases the sales of imposing product A, as the network effect dominates the price effect. Both indirect and direct effects of the externality on welfare are negative, indicating that welfare is unambiguously reduced by enlargement of the negative externality above its equilibrium level. When demand for A is low enough (i.e., $\theta < -\frac{1}{2}$), the indirect effect of the externality on welfare turns positive, as an increase in its size reduces the sales of A. But if A's demand is not *so* low, as at $\theta = -0.75$, the negative direct effect overwhelms the positive indirect effect so that an increase in the size of the negative externality at equilibrium reduces welfare overall. However, if demand for A is *very* low, as at $\theta = -1$, the overall effect on welfare of increasing the externality is positive: the positive indirect effect overwhelms the negative direct effect, so that though each unit of the imposing product causes more harm, the number of units sold is reduced sufficiently to improve welfare. Thus the numerical example illustrates the astonishing finding that a public policy to enlarge product A's negative externality improves welfare if demand for A is below a certain threshold.

IV. CONCLUSION

This paper has considered the effects on market equilibrium and public policy outcomes of negative externalities occurring in a competitive, differentiated product market context. It has used a simple model to explore how a selective negative externality – one that affects consumers of competing products more than consumers of the imposing product – impacts prices and consumer choices observed in equilibrium. While the outcome that selective externalities have competitive effects is intuitive on its face, its implications are surprising. When demand for the imposing product is relatively low, the usual welfare result relating to negative externalities is turned on its head, and policy interventions to enlarge the size of the negative externality improve welfare.

The findings suggest that public policies to address negative externalities of consumption must account for competitive effects. Policymakers must consider whether the externality to be remedied appears to be nonuser-specific, either entirely or in relative terms. If it is, both direct external effects and the indirect (competitive) effects of the externality must be taken into account in establishing the best mechanism for remediation. The model's results demonstrate that the optimal approach might in certain cases be the opposite of what would be indicated based only on direct external effects. Still, further work is needed to work out the details with respect to applied policies. Consider, for example, the design of Pigouvian taxes, which have been held up as a way to address the welfare losses associated with negative externalities (e.g., Kopczuk [2003]). What does an optimal Pigouvian tax look like for a selective negative externality? In view of our public policy result, might a subsidy sometimes called be for, rather than a tax?

The model here has offered an exploratory analysis, and there is still much to do. The implications of selective negative externalities in alternative contexts to those represented in the model should be considered. The frontiers of the analysis should be expanded to include, for example, the implications of alternative market structures to duopoly, consumer multi-homing, and competition among multiple externality-imposing goods.

Other relevant issues relate to the differences between network externalities arising from selective (nonuser-specific) negative externalities and those (the "conventional" kind) that arise from user-specific positive consumption externalities. For example, the former differ from the latter in that they are not characterized by mutuality: that is, the network effect stemming from a nonuser-specific negative externality relates not to a benefit shared among compatible users, but to a cost "network" users impose unilaterally on "incompatible" users. The implications of this for compatibility outcomes must be examined. Does the lack of mutuality affect the extent to which private incentives for compatibility diverge from socially optimal incentives? And how should public policy towards compatibility be modified to deal with this particular kind of network externality? Nagler [2008] considers some of these issues.

Finally, the competitive effects of externalities should be estimated empirically. Specifically, the CESE, identified in Propositions 1 and 4, might form the basis for a measurable "external cost elasticity of demand." One might expect zero demand elasticity for conventional negative externalities, and a significant positive or negative elasticity where externalities are competitively relevant.

APPENDIX

Analytical Solution to the Equilibrium, Externality-Imposing Market Product versus Free Product. We begin by substituting the solution to the second-stage first-order condition, $p_A^* = \theta + \frac{t}{2}$, into (5) to obtain a partial reduced-form for Q_A^*:

$$Q_A^* = \frac{\frac{t}{2} + \theta}{2t - \sigma\lambda} \tag{A1}$$

Let $h_{MF}\left(\lambda \middle| \underline{\lambda}, t, k, \sigma\right)$ be defined as the marginal revenue to firm A from increasing λ. Then, λ^* is the value of λ that generates the greatest profit for A from among those values that satisfy $h_{MF}\left(\lambda\right) = C'\left(\lambda\right)$ and $h_{MF}'\left(\lambda\right) < C''\left(\lambda\right)$, that is, it is the most profitable local maximum. To identify and compare candidate equilibria, we must specify the characteristics of $h_{MF}\left(\lambda\right)$ in all relevant cases. $h_{MF}\left(\lambda\right)$ is given, for an interior solution, by the left-hand side of (6). With substitutions from (A1) and $p_A^* = \theta + \frac{t}{2}$, (A2) $$h_{MF}\left(\lambda\right)_{Q_A,Q_B\in(0,1)} \equiv \frac{\sigma}{\left(2t-\sigma\lambda\right)^2}\left(\theta+\frac{t}{2}\right)^2$$

Analysis reveals that this expression is increasing and convex on $\lambda < \frac{2t}{\sigma}$, and that it approaches ∞ as λ approaches $\frac{2t}{\sigma}$.

Three corner solutions must also be considered. First, note that $p_A, Q_A = 0$ results when $\theta < -\frac{t}{2}$ and $\lambda < \frac{2t}{\sigma}$. It may be observed from (A1) and $p_A^* = \theta + \frac{t}{2}$ that as θ approaches $-\frac{t}{2}$ from above, then, assuming $\lambda < \frac{2t}{\sigma}$, both p_A and Q_A approach 0. This reflects the intuition that product A becomes unprofitable to sell as underlying demand for the product becomes sufficiently small. Meanwhile, (A2) shows that $h_{MF}\left(\lambda\right)$ approaches 0 as θ approaches $-\frac{t}{2}$ (also assuming $\lambda < \frac{2t}{\sigma}$): firm A simply obtains no revenue from increasing λ when it is not going to sell any product anyway. So $h_{MF}\left(\lambda\right) = 0$ is associated with a $Q_A = 0$ corner.

Second, (A1) shows that when $\theta > -\frac{t}{2}$, increases to λ increase Q_A without bound; thus $Q_A = 1$ will be reached when λ is large enough. Setting (A1) equal to 1 yields the threshold level, $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$. Now, the marginal revenue from increasing λ when $Q_A = 1$ equals the marginal effect of λ on A's price, because A's share of the market is held constant (at 1). Using $p_A = 2\theta - t + \sigma\lambda$, which arises from setting (5) equal to 1, $h_{MF}\left(\lambda\right) = \sigma$ at $Q_A = 1$.

Finally, when $\theta < -\frac{t}{2}$, another $Q_A = 1$ solution exists corresponding to A setting $\lambda > \frac{2t}{\sigma}$. In this case, with $p_A = 2\theta - t + \sigma\lambda$, consumers choosing product A will not have an incentive to switch to B. As in the previous $Q_A = 1$ case, $h_{MF}\left(\lambda\right) = \sigma$ holds.

The process of determining the equilibrium using $h_{MF}\left(\lambda\right)$ is illustrated graphically in the four panels of Figure 3. Which case in Figure 3 is relevant depends upon the relevant case of

$h_{MF}(\lambda)$. Candidate equilibria are those where $C'(\lambda)$ intersects $h_{MF}(\lambda)$ from below. Given the cases identified with respect to $h_{MF}(\lambda)$, there will be a minimum of one, and maximum of two, candidate equilibria. Figure 3(a) displays a case with one candidate equilibrium; the single intersection of $h_{MF}(\lambda)$ with $C'(\lambda)$ gives the value of λ^*. Figure 3(c) displays a case with two candidate equilibria. Here, one must compare the relative sizes of shaded areas "I" and "II" to determine which of the equilibria maximizes A's profits: if "II" is larger, then it is the corner solution with $Q_A = 1$ and $\lambda = \tilde{\lambda}$; but if "II" is smaller, then it is the interior solution with $\lambda = \underset{\sim}{\lambda}$.

(Place Figures 3(a), 3(b), 3(c), and 3(d) together approximately here.)

First consider the case of $\theta > -\frac{t}{2}$, which implies $h_{MF}(\lambda)$ is convex on $\lambda < \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$, and flat at $h_{MF}(\lambda) = \sigma$ on $\lambda > \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$. Three sub-cases obtain, depending upon whether $h_{MF}(\lambda)$ crosses $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$ below or above $C'(\lambda)$, and if the latter, whether $h_{MF}'(\lambda) < C''(\lambda)$ at $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$. Suppose $h_{MF}(\lambda)$ crosses $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$ below $C'(\lambda)$. Substituting $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$ into $C'(\lambda) = k(\lambda - \underline{\lambda})$, and setting $C'(\lambda) > h_{MF}(\lambda) = \sigma$, yields $\theta < \frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{k}$. Figure 3(a) illustrates this case: the equilibrium consists of a single interior solution, implied by $\lambda^* < \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$, with some consumers choosing A and some B.

If instead $\theta > \frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{k}$, then $h_{MF}(\lambda)$ crosses $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$ above $C'(\lambda)$, so one must check whether $h_{MF}'(\lambda) < C''(\lambda)$ at $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$, to wit, $\theta > \frac{2\sigma^2}{k} - \frac{t}{2}$. If this holds, then the $Q_A = 1$ threshold occurs *before* the first intersection of $h_{MF}(\lambda)$ and $C'(\lambda)$; as Figure 3(b) illustrates, all consumers choose A in this unambiguous equilibrium corresponding to a relatively low $\lambda^* > \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$. Now if instead $\theta < \frac{2\sigma^2}{k} - \frac{t}{2}$, the relevant case, as illustrated by Figure 3(c), involves a "triple crossing" whereby firm A could set $\lambda = \underset{\sim}{\lambda}$ such that some consumers choose A while others choose B, or $\lambda = \tilde{\lambda}$ to induce all consumers to choose A.

But this latter option implies tipping the market. To preclude tipping, it is sufficient to rule out the sub-case entirely by imposing the "no-tipping" condition that σ satisfy $\frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{k} > \frac{2\sigma^2}{k} - \frac{t}{2}$, or

$$\text{(A3)} \qquad \sigma < \sigma_{NTF} \equiv \frac{k}{6}\left[\sqrt{\underline{\lambda}^2 + \frac{24t}{k}} - \underline{\lambda}\right]$$

Under (A3), $\theta > \frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{k}$ implies $\theta > \frac{2\sigma^2}{k} - \frac{t}{2}$, yielding an unambiguous equilibrium in which all consumers choose A.

Now consider $\theta < -\frac{t}{2}$. As noted previously, $h_{MF}(\lambda) = 0$ when $\lambda < \frac{2t}{\sigma}$, while $h_{MF}(\lambda) = \sigma$ when $\lambda > \frac{2t}{\sigma}$. In this case, illustrated in Figure 3(d), firm A could set $\lambda = \underline{\lambda}$ such that all consumers to choose B, or it could set $\lambda = \tilde{\lambda}$, potentially tipping the market. But, in fact, (A3) implies no tipping when $\theta < -\frac{t}{2}$ as well. To see this, note that (A3) implies $C'\left(\frac{2t}{\sigma}\right) > \sigma$. That $C'\left(\frac{2t}{\sigma}\right) > \sigma$ implies no tipping when $\theta < -\frac{t}{2}$ can be observed by picturing in Figure 3(d) a horizontal line for $h_{MF}(\lambda) = \sigma$ that intersects $\lambda = \frac{2t}{\sigma}$ below $C'(\lambda)$. In this case, there is only a single crossing and only one candidate equilibrium at $\lambda = \underline{\lambda}$. Thus (A3) assures an unambiguous equilibrium in which all consumers choose B when $\theta < -\frac{t}{2}$.

Note that we have shown that $\lambda < \frac{2t}{\sigma}$ also follows from no-tipping condition (A3).

Analytical Solution to the Equilibrium, Externality-Imposing Market Product versus Non-Imposing Market Product. As in the free-B case, we begin by obtaining a partial reduced-form for Q_A^* (here, substituting (11) into the first equation in (10)):

(A4) $$Q_A = \frac{t + \frac{2}{3}\theta - \frac{1}{3}\sigma\lambda}{2t - \sigma\lambda}$$

We proceed by specifying the characteristics of $h_{MM}\left(\lambda \middle| \underline{\lambda}, t, k, \sigma\right)$, the marginal revenue to A from increasing λ when there are two market products, in all relevant cases. For an interior solution, $h_{MM}\left(\lambda\right)$ is given by the left-hand side of (12). With substitutions from (11) and (A4), we obtain

(A5) $$h_{MM}\left(\lambda\right)_{Q_A, Q_B \in (0,1)} \equiv \frac{4}{9}\frac{\sigma}{\left(2t-\sigma\lambda\right)^2}\left(\theta + \frac{t}{2}\right)^2 - \frac{\sigma}{9} = \frac{4}{9}h_{MF}\left(\lambda\right) - \frac{\sigma}{9}$$

As a linear transformation of h_{MF}, the function h_{MM} is similarly increasing and convex on $\lambda < \frac{2t}{\sigma}$; however, the negative additive term indicates that h_{MM} could be negative; this implies the possibility of abatement of the externality (i.e., $\lambda^* < \underline{\lambda}$) in equilibrium.

Analysis of (10), (11), and (A4) reveals that the same three corner solutions found in the free-B case exist here, with minor modifications. First, $p_A, Q_A = 0$ results when $\theta < -\frac{t}{2}$ and $\lambda < \frac{2t}{\sigma}$, with the additional requirement that $\lambda \geq \frac{2\theta + 3t}{\sigma}$; at this corner, $h_{MM}\left(\lambda\right) = 0$. Second, $p_B, Q_B = 0$ (hence $Q_A = 1$) results when $\theta > -\frac{t}{2}$ and $\lambda \geq \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$, and this outcome corresponds to $h_{MM}\left(\lambda\right) = \sigma$. However, unlike the free-B case, notice that as λ approaches $\frac{3t}{2\sigma} - \frac{\theta}{\sigma}$ from below, $h_{MM}\left(\lambda\right)$ approaches $\frac{\sigma}{3}$. This is because the marginal revenue from increasing λ is reduced relative to free-B by the negative effect of λ on price and its positive effect on the price differential. The price effects are gone when $p_B, Q_B = 0$, hence a discontinuity in the marginal revenue function $h_{MM}\left(\lambda\right)$ at $\frac{3t}{2\sigma}-\frac{\theta}{\sigma}$. Finally, $\theta<-\frac{t}{2}$ and $\lambda>\frac{2t}{\sigma}$ together yield a corner outcome of $p_B, Q_B=0$ corresponding to $h_{MM}\left(\lambda\right)=\sigma$.

We follow essentially the same technique as in the free-B case to determine the equilibrium using $h_{MM}\left(\lambda\right)$. The outcome depends upon the relevant case of $h_{MM}\left(\lambda\right)$, the location of the threshold for the $Q_A=1$ corner solution, and parameters determining A's preference between candidate equilibria.

First, consider the case of $\theta>-\frac{t}{2}$. The analysis is complicated slightly relative to the free-B case by the discontinuity in $h_{MM}\left(\lambda\right)$ at $\lambda=\frac{3t}{2\sigma}-\frac{\theta}{\sigma}$, which requires us to consider three sub-cases with respect to the position of $h_{MM}\left(\lambda\right)$ relative to $C'\left(\lambda\right)$ at $\lambda=\frac{3t}{2\sigma}-\frac{\theta}{\sigma}$. Suppose first that $h_{MM}\left(\lambda\right)$ is below $C'\left(\lambda\right)$ *on both sides of* $\lambda=\frac{3t}{2\sigma}-\frac{\theta}{\sigma}$. This implies $C'\left(\frac{3t}{2\sigma}-\frac{\theta}{\sigma}\right)>\sigma$, or $\theta<\frac{3t}{2}-\underline{\lambda}\sigma-\frac{\sigma^2}{k}$. The outcome is a trivial variation on Figure 3(a), and we have a confirmed interior solution with some consumers choosing A and some choosing B. Now suppose instead that $h_{MM}\left(\lambda\right)$ is *above* $C'\left(\lambda\right)$ on both sides of $\lambda=\frac{3t}{2\sigma}-\frac{\theta}{\sigma}$. This implies $C'\left(\frac{3t}{2\sigma}-\frac{\theta}{\sigma}\right)<\frac{\sigma}{3}$, or $\theta>\frac{3t}{2}-\underline{\lambda}\sigma-\frac{\sigma^2}{3k}$. The exact outcome depends upon whether $h_{MM}'\left(\lambda\right)<C''\left(\lambda\right)$ at $\lambda=\frac{3t}{2\sigma}-\frac{\theta}{\sigma}$; let us assume for the moment that this is true. Then, the equilibrium is a trivial variation on Figure 3(b), yielding an unambiguous corner solution in which all consumers choose A. Now consider the third possibility, $C'\left(\frac{3t}{2\sigma}-\frac{\theta}{\sigma}\right)\in\left(\frac{\sigma}{3},\sigma\right)$, such that $C'\left(\lambda\right)$ crosses $h_{MM}\left(\lambda\right)$ at its discontinuity. Figure 4 illustrates. Whether there is an interior solution at $\lambda^*=\underset{\sim}{\lambda}$ or corner solution at $\lambda^*=\tilde{\lambda}$ depends upon the relative size of shaded areas "I" and "II." Since an increase in θ shifts

$h_{MM}(\lambda)$ upward, it follows that there exists $\theta_H \in \left(\frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{k}, \frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{3k}\right)$, such that $\theta > \theta_H$ implies a corner solution whereas $\theta < \theta_H$ implies an interior solution.

(Place Figure 4 approximately here.)

Returning to the sub-case where $\theta > \frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{3k}$, notice that $\theta > \frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{3k}$ raises the possibility of a triple crossing between h_{MM} and $C'(\lambda)$, similar to the tipping situation illustrated in Figure 3(c) with respect to the free-B case, but with a discontinuity in h_{MM} above the second intersection. This scenario corresponds to $h_{MM}'(\lambda) > C''(\lambda)$ at $\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$, or $\theta < \frac{8}{9}\frac{\sigma^2}{k} - \frac{t}{2}$. We rule out the scenario entirely by imposing the no-tipping condition that σ satisfy $\frac{3t}{2} - \underline{\lambda}\sigma - \frac{\sigma^2}{3k} > \frac{8\sigma^2}{9k} - \frac{t}{2}$, or

(A6) $$\sigma < \frac{9k}{22}\left[\sqrt{\underline{\lambda}^2 + \frac{88t}{9k}} - \underline{\lambda}\right]$$

Now consider $\theta < -\frac{t}{2}$. First, we are able to rule out a corner solution corresponding to $\lambda > \frac{2t}{\sigma}$, applying to our current no-tipping condition (A6) the same logic we applied in the free-B case to (A3). This restricts us to an equilibrium in which $\lambda < \frac{2t}{\sigma}$. Now, as noted above, $h_{MM} = 0$ for $\lambda > \frac{3t+2\theta}{\sigma}$, and inspection of (A5) indicates that $h_{MM} < 0$ for $\lambda < \frac{3t+2\theta}{\sigma}$. Thus, $h_{MM}(\lambda)$ is kinked, so that three possible sub-cases obtain, as indicated by the three panels of Figure 5. These depend upon whether $C'(\lambda) > h_{MM}(\lambda)$ at $\lambda = \frac{3t+2\theta}{\sigma}$, and, if not, then whether $h_{MM}'(\lambda) < C''(\lambda)$ or not at $\lambda = \frac{3t+2\theta}{\sigma}$.

(Place Figures 5(a), 5(b), and 5(c) together approximately here.)

Suppose first that $C'(\lambda) > h_{MM}(\lambda)$ at $\lambda = \frac{3t+2\theta}{\sigma}$. This sub-case corresponds to $\theta > \frac{\sigma\underline{\lambda}-3t}{2}$. Figure 5(a) illustrates: the equilibrium consists of a single interior solution corresponding to $\lambda^* = \underset{\sim}{\lambda} < \underline{\lambda}$, with some consumers choosing A and some B.

If, instead, $\theta < \frac{\sigma\underline{\lambda}-3t}{2}$, implying $C'(\lambda) < h_{MM}(\lambda)$ at $\lambda = \frac{3t+2\theta}{\sigma}$, there are two possibilities to consider. If $h_{MM}'(\lambda) < C''(\lambda)$ at $\lambda = \frac{3t+2\theta}{\sigma}$ (that is, $\theta < -\frac{8}{9}\frac{\sigma^2}{k} - \frac{t}{2}$), then as illustrated by Figure 5(b), a corner solution obtains in which $\lambda^* = \underline{\lambda} > \frac{3t+2\theta}{\sigma}$ and all consumers choose B. But suppose instead that $h_{MM}'(\lambda) > C''(\lambda)$ at $\lambda = \frac{3t+2\theta}{\sigma}$ (that is, $\theta > -\frac{8}{9}\frac{\sigma^2}{k} - \frac{t}{2}$). As Figure 5(c) illustrates, this introduces the possibility of a triple crossing, such that the equilibrium will occur at $\underline{\lambda}$ or $\underset{\sim}{\lambda}$, depending upon the relative size of shaded areas "I" and "II." We rule this scenario out by imposing a no-tipping condition that σ satisfy $\frac{\sigma\underline{\lambda}-3t}{2} < -\frac{8}{9}\frac{\sigma^2}{k} - \frac{t}{2}$, or

(A7) $$\sigma < \frac{9k}{32}\left[\sqrt{\underline{\lambda}^2 + \frac{128t}{9k}} - \underline{\lambda}\right]$$

Combining (A6) and (A7), we obtain a single no-tipping condition for the market product vs. market product case,

(A8) $$\sigma < \sigma_{NTM} \equiv \min\left\{\frac{9k}{32}\left[\sqrt{\underline{\lambda}^2 + \frac{128t}{9k}} - \underline{\lambda}\right], \frac{9k}{22}\left[\sqrt{\underline{\lambda}^2 + \frac{88t}{9k}} - \underline{\lambda}\right]\right\}$$

Proof of Proposition 2. Substituting and integrating in (7) yields

(A9) $$W = \Pi_A + (\sigma\lambda - t)Q_A^2 + (2\theta - p_A + t - \lambda)Q_A + v - \theta - \frac{t}{2}$$

Differentiate with respect to λ, using (5), and recognizing that $\frac{\partial p_A}{\partial\lambda} = 0$:

$$W = \Pi_A + (\sigma\lambda - t)Q_A^2 + (2\theta - p_A + t - \lambda)Q_A + v - \theta - \frac{t}{2}$$
$$\Rightarrow \frac{\partial W}{\partial\lambda} = \frac{\partial\Pi_A}{\partial\lambda} + \sigma Q_A^2 + 2(\sigma\lambda - t)Q_A\left[\frac{\sigma Q_A}{2t-\sigma\lambda}\right] - Q_A + (2\theta - p_A + t - \lambda)\left[\frac{\sigma Q_A}{2t-\sigma\lambda}\right]$$

Because this expression is evaluated at the profit maximum, $\frac{\partial \Pi_A}{\partial \lambda} = 0$. Thus, using $p_A^* = \theta + \frac{t}{2}$ and (A1) and factoring:

$$\frac{\partial W}{\partial \lambda} = \sigma Q_A^2 - Q_A + 2(\sigma\lambda - t)Q_A\left[\frac{\sigma Q_A}{2t-\sigma\lambda}\right] + (2\theta - p_A + t - \lambda)\left[\frac{\sigma Q_A}{2t-\sigma\lambda}\right]$$

$$= \frac{\sigma\left(\frac{t}{2}+\theta\right)^2}{(2t-\sigma\lambda)^2} - \frac{\frac{t}{2}+\theta}{2t-\sigma\lambda} + \frac{2\sigma(\sigma\lambda - t)\left(\frac{t}{2}+\theta\right) + \sigma(2t-\sigma\lambda)\left(2\theta - \left(\frac{t}{2}+\theta\right) + t - \lambda\right)}{2t-\sigma\lambda}\left[\frac{Q_A}{2t-\sigma\lambda}\right]$$

$$= \frac{2t\left(\frac{t}{2}+\theta\right)}{(2t-\sigma\lambda)^3}\left[\sigma\left(\theta+\frac{t}{2}\right) - (2t-\sigma\lambda)\right]$$

Based on (A1), the expression in brackets is positive if and only if $Q_A > \frac{1}{\sigma}$. Since $0 \le \sigma \le 1$, $0 < Q_A < 1$ implies the expression is negative, which in turn implies that $\frac{\partial W}{\partial \lambda}$ takes the sign of $-\left(\frac{t}{2}+\theta\right)$ (since $2t - \sigma\lambda > 0$ also follows from $0 < Q_A < 1$). Moreover, $0 < Q_A < 1$ implies $\theta > -\frac{t}{2}$, hence $-\left(\frac{t}{2}+\theta\right)$ negative and $\frac{\partial W}{\partial \lambda} < 0$. ■

Propositions 1 and 2 – General Case. Per the discussion in the text, assume general utility functions

$$U_A = v + \theta + f^A(\lambda, Q_A) - t(1-j) - p_A$$

$$U_B = v - \theta + f^B(\lambda, Q_A) - tj$$

where $f^A(0,Q_A) = f^B(0,Q_A) = 0 \;\; \forall Q_A \in (0,1)$, $f^A(\lambda,0) = f^B(\lambda,0) = 0 \;\; \forall \lambda > 0$ and

$$f_\lambda^B(\lambda,Q_A) < f_\lambda^A(\lambda,Q_A) < 0 \;\; \forall \lambda > 0,\, Q_A \in (0,1)$$

$$f_{Q_A}^B(\lambda,Q_A) < f_{Q_A}^A(\lambda,Q_A) < 0 \;\; \forall \lambda > 0,\, Q_A \in (0,1)$$

$$f_{\lambda Q_A}^B(\lambda,Q_A) < f_{\lambda Q_A}^A(\lambda,Q_A) < 0 \;\; \forall \lambda > 0,\, Q_A \in (0,1)$$

Define $h(\lambda, Q_A) \equiv f^A - f^B$. It follows that $h(\lambda, Q_A) > 0 \;\; \forall \lambda > 0, Q_A \in (0,1)$; and $h_\lambda > 0$, $h_{Q_A} > 0$, and $h_{\lambda Q_A} > 0$. The reservation price of A may now be written

$$\Psi_j^{MF} \equiv 2\theta + h(\lambda, Q_A) - t(1-2j)$$

and we require $h_{Q_A} < 2t$ for stability.

Define j^* as the threshold *j,* that is, $\Psi_{j^*}^{MF} = p_A$. Then we have

$$1 - Q_A \equiv j^* = \frac{p_A - 2\theta - h(\lambda, Q_A)}{2t} + \frac{1}{2} \tag{A10}$$

We cannot solve for the demand function for product A in closed form, so we develop the following implicit function, taking p_A as given,

$$g(\lambda, Q_A) = \frac{p_A - 2\theta - h(\lambda, Q_A)}{2t} - \frac{1}{2} + Q_A$$

where $g(\lambda, Q_A) = 0$ yields $Q_A(p_A, t, \theta, \lambda)$. Using the implicit function rule, we obtain

$$\frac{\partial Q_A}{\partial p_A} = -\frac{g_{p_A}}{g_{Q_A}} = -\frac{1}{2t - h_{Q_A}}; \quad \left.\frac{\partial Q_A}{\partial \lambda}\right|_{\bar{p}_A} = -\frac{g_\lambda}{g_{Q_A}} = \frac{h_\lambda}{2t - h_{Q_A}}$$

where the second equation assumes p_A is held constant.

Because the game is two-stage, however, we must account for the effect of first-stage changes in λ on firm A's second-stage choice of p_A, that is, $\frac{\partial p_A}{\partial \lambda}$. We calculate this using A's first-order condition for price:

$$\Gamma(p_A, t, \theta, \lambda) \equiv Q_A(p_A, t, \theta, \lambda) + p_A \frac{\partial Q_A}{\partial p_A} = 0$$

Again using the implicit function rule

$$\frac{\partial p_A}{\partial \lambda} = -\frac{\frac{\partial \Gamma}{\partial \lambda}}{\frac{\partial \Gamma}{\partial p_A}} = -\frac{\frac{\partial Q_A}{\partial \lambda} + p_A \frac{\partial^2 Q_A}{\partial \lambda \partial p_A}}{2\frac{\partial Q_A}{\partial p_A} + p_A \frac{\partial^2 Q_A}{\partial p_A^2}} \tag{A11}$$

where

$$\frac{\partial^2 Q_A}{\partial\lambda\partial p_A} = -\frac{h_{\lambda Q_A} + h_{Q_A Q_A}\frac{\partial Q_A}{\partial\lambda}}{\left(2t-h_{Q_A}\right)^2} = -\frac{h_{\lambda Q_A} + h_{Q_A Q_A}\left(\frac{h_\lambda}{2t-h_{Q_A}}\right)}{\left(2t-h_{Q_A}\right)^2} = -\frac{h_{\lambda Q_A}}{\left(2t-h_{Q_A}\right)^2} - \frac{h_{Q_A Q_A} h_\lambda}{\left(2t-h_{Q_A}\right)^3}$$

$$\frac{\partial^2 Q_A}{\partial p_A^2} = -\frac{h_{Q_A Q_A}}{\left(2t-h_{Q_A}\right)^2}\cdot\frac{\partial Q_A}{\partial p_A} = \frac{h_{Q_A Q_A}}{\left(2t-h_{Q_A}\right)^3}$$

Solving into (A11) yields

(A12) $$\frac{\partial p_A}{\partial\lambda} = h_\lambda + \left(2t-h_{Q_A}\right)\frac{h_\lambda\left(2t-h_{Q_A}\right) + p_A h_{\lambda Q_A}}{p_A h_{Q_A Q_A} - 2\left(2t-h_{Q_A}\right)^2}$$

Thus,

$$\frac{\partial Q_A}{\partial\lambda} = \left.\frac{\partial Q_A}{\partial\lambda}\right|_{\bar{p}_A} + \frac{\partial Q_A}{\partial p_A}\frac{\partial p_A}{\partial\lambda} = -\frac{h_\lambda\left(2t-h_{Q_A}\right) + p_A h_{\lambda Q_A}}{p_A h_{Q_A Q_A} - 2\left(2t-h_{Q_A}\right)^2}$$

Clearly $\frac{\partial Q_A}{\partial\lambda} > 0$ when $h_{Q_A Q_A} \le 0$, and also when $h_{Q_A Q_A} > 0$ but small. But $\frac{\partial Q_A}{\partial\lambda} \to \infty$ as $h_{Q_A Q_A}$ grows, so a limit on the convexity of *h* is required as an additional "no-tipping" restriction in this general case. Subject to this, $\frac{\partial Q_A}{\partial\lambda} > 0$ is assured, affirming the result in Proposition 1.

To prove Proposition 2 generally, we substitute the general utility functions into (7) and integrate to obtain

$$W = \Pi_A - tQ_A^2 + \left[2\theta + h\left(\lambda,Q_A\right) + t - p_A\right]Q_A + v - \theta + f^B\left(\lambda,Q_A\right) - \frac{t}{2}$$

Differentiating with respect to λ,

$$\frac{\partial W}{\partial\lambda} = \frac{\partial\Pi_A}{\partial\lambda} + \left[2\theta + h\left(\lambda,Q_A\right) + t - p_A - 2tQ_A + h_{Q_A}Q_A\right]\frac{\partial Q_A}{\partial\lambda} + Q_A\left(h_\lambda - \frac{\partial p_A}{\partial\lambda}\right) + \left(f_\lambda^B + f_{Q_A}^B\frac{\partial Q_A}{\partial\lambda}\right)$$

Now, we use $\frac{\partial\Pi_A}{\partial\lambda} = 0$ and make substitutions from (A10) and (A12) to obtain

$$\frac{\partial W}{\partial\lambda} = \left(2tQ_A + f_{Q_A}^B\right)\frac{\partial Q_A}{\partial\lambda} + f_\lambda^B$$

The expression in parentheses is signed ambiguously. Suppose first that it is negative. Then $\frac{\partial Q_A}{\partial \lambda} > 0$ is sufficient for $\frac{\partial W}{\partial \lambda} < 0$, so Proposition 2 follows from the proof of Proposition 1. If, instead, the expression is positive, then the "no-tipping" restriction on the convexity of *h* is not sufficient for $\frac{\partial W}{\partial \lambda} < 0$, as there exists $h_{Q_A Q_A} > 0$ such that $\frac{\partial Q_A}{\partial \lambda} > 0$ and finite, but such that $\frac{\partial W}{\partial \lambda} \geq 0$. But as $\frac{\partial W}{\partial \lambda} < 0$ when $h_{Q_A Q_A} = 0$, then it is also true that $\frac{\partial W}{\partial \lambda} < 0$ for $h_{Q_A Q_A} < 0$ (i.e., when $\frac{\partial Q_A}{\partial \lambda}$ is smaller). We may assure $\frac{\partial W}{\partial \lambda} < 0$, then, by requiring $h_{Q_A Q_A} \leq 0$ or sufficiently limiting the convexity of *h*. This affirms Proposition 2. ■

Proof of Proposition 4 and Corollary 4.1. Using the expression for Q_A in (10), we differentiate with respect to λ, decomposing as follows:

$$\frac{\partial Q_A}{\partial \lambda} + \frac{\partial Q_A}{\partial (p_A - p_B)} \frac{\partial (p_A - p_B)}{\partial \lambda} = \frac{\sigma Q_A}{2t - \sigma\lambda} - \frac{1}{3}\frac{\sigma}{2t - \sigma\lambda}$$

Thus the direct effect (first term) is positively signed and the indirect effect (second term) negatively signed. Summing the two terms together yields $\dfrac{\sigma\left(Q_A - \frac{1}{3}\right)}{2t - \sigma\lambda}$, which is positively signed if and only if $Q_A > \frac{1}{3}$. Using (10), we see that this condition is equivalent to $\theta > -\frac{t}{2}$. ■

Proof of Proposition 5. Given that (15) is identically equal to (7), we may proceed using (A9). Differentiating with respect to λ obtains:

(A13) $$\frac{\partial W}{\partial \lambda} = \frac{\partial \Pi_A}{\partial \lambda} + \sigma Q_A^2 + 2(\sigma\lambda - t) Q_A \frac{\partial Q_A}{\partial \lambda} - \left(1 + \frac{\partial p_A}{\partial \lambda}\right) Q_A + (2\theta - p_A + t - \lambda)\frac{\partial Q_A}{\partial \lambda}$$

Because this expression is evaluated at the profit maximum, $\frac{\partial \Pi_A}{\partial \lambda} = 0$. When $\theta = -\frac{3t}{2} + \frac{\sigma\lambda}{2}$, then $\frac{\partial W}{\partial \lambda} = 0$, because $Q_A = 0$, hence $\frac{\partial Q_A}{\partial \lambda} = 0$. To evaluate $\frac{\partial W}{\partial \lambda}$ on $0 < Q_A < 1$, observe that

$\frac{\partial p_A}{\partial \lambda} = -\frac{1}{3}\sigma$ and $\frac{\partial Q_A}{\partial \lambda} = \dfrac{\sigma\left(Q_A - \frac{1}{3}\right)}{2t - \sigma\lambda}$. Substituting these expressions into (A13), using (10) and (11) and factoring, yields

$$\text{(A14)}\quad \frac{\partial W}{\partial \lambda} = \frac{\left(\frac{2}{3}\sigma - 1\right)\left(2t - \sigma\lambda\right)^3 + 2t\left(2\theta + t\right)\left(2t - \sigma\lambda\right)\left(\frac{1}{3}\sigma - 1\right) + \frac{1}{3}\sigma\left(4t - \sigma\lambda\right)\left(2\theta + t\right)^2}{3\left(2t - \sigma\lambda\right)^3}$$

Since $\lambda < \frac{2t}{\sigma}$ on $0 < Q_A < 1$, hence on $\theta \in \left(-\frac{3t}{2} + \frac{\sigma\lambda}{2}, -\frac{t}{2}\right]$, it follows that $\frac{\partial W}{\partial \lambda} < 0$ at $\theta = -\frac{t}{2}$. If we can show that $\frac{\partial W}{\partial \lambda} > 0$ at θ greater than but arbitrarily close to $-\frac{3t}{2} + \frac{\sigma\lambda}{2}$, then we have proven the proposition. Substituting $\theta = -\frac{3t}{2} + \frac{\sigma\lambda}{2}$ and $\lambda = \underline{\lambda}$ into (A14) yields $\frac{\partial W}{\partial \lambda} = \frac{\sigma}{3} + \frac{\sigma\lambda}{3(2t - \sigma\lambda)} > 0$. ■

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[1] Cognitive dissonance may be described as the internal conflict that results when an individual receives information that contradicts basic ego-supporting beliefs. For example, an individual tends to think, "I am a nice person who would never do something to irritate or harm an undeserving person." So if one engages in an activity that one previously found irritating when others did it, internal stress might be averted by changing one's attitude so as to view the activity as less noisome. Evidence of attitude change as a reaction to dissonance-provoking situations is provided by Festinger and Carlsmith [1959], Davis and Jones [1960], and Glass [1964]. For an economic discussion of cognitive dissonance, see Akerlof and Dickens [1982].

[2] Along these lines, Stiglitz [2008] pointed out a growing stigma for banks that had not elected to join the U. S. Treasury's bailout plan during October 2008 as other banks joined.

[3] It is important to distinguish this phenomenon from positional goods (and activities), discussed by Frank [1985, 1991], with respect to which adopters benefit by obtaining improved position in a hierarchy. One example offered by Frank [1999, pp. 154-155] is the practice of "redshirting," whereby parents start their children a year later in kindergarten to give them a relative academic advantage over other children in their kindergarten class. For such an activity or good to be *purely* positional, it must be that the adopter does not impose an incremental stigma on those lower in the hierarchy, so benefits accrue *only* to relative position.

[4] See, for example, White [2004] with respect to SUVs and Ayres and Levitt [1998] with respect to visible anti-theft devices.

[5] Previous analyses of network externalities (e.g., Economides [1996]) have not distinguished effects of the number of users on the *utility of users* from their effects on *users' opportunity costs*, which are more directly responsible for use decisions. The reason is that previous work has focused on selective external

effects by users of a product on *other users*, such that effects on utility and opportunity costs are perfectly negatively correlated. In contrast, the present paper considers external costs selectively imposed on *non-users*, whereby the opportunity cost of the "network" good is decreased without affecting the utility the consumer obtains from that good. It is therefore appropriate that we define a network externality as involving the dependence of the opportunity cost for the network good, rather than utility from its use, on the number of users. The new definition is general enough to encompass network effects based on both user-specific and nonuser-specific positive and negative externalities. (I am indebted to Christiaan Hogendorn for his help in articulating this issue.)

[6] External costs can often be manipulated through product design. SUVs generally have high, stiff front ends that increase the damage done to the vehicles with which they collide; this effect could be undone through various design changes (see, e.g., Bradsher [2002], Latin and Kasolas [2002]). Cigarettes could be manufactured to give off more or less smoke from the lit end, and the amount of smoke and noise emitted by gasoline-powered outdoor equipment could similarly be altered. Meanwhile, the size of the external costs that consumers *perceive* might be manipulated using marketing messages. For instance, calling greater attention to how imposing a particular SUV is might convince consumers that it is more dangerous to other motorists. (See Bradsher [2002] for examples of intimidating SUV advertisements.) By advertising, "Don't be the last programmer in the market to get one," a purveyor of computer programming certifications might enlarge perceptions of the stigma imposed on non-adopters by incremental adoptions.

[7] In fact, the term "crash test compatibility" is sometimes associated with the first question in the case of cars versus SUVs. See Bradsher [2002].

[8] See footnote 6 *supra*.

[9] Other scenarios described in the introduction do not fit as neatly into the modified duopoly framework proposed by this model. For example, the motor vehicle industry consists of firms that produce both cars and SUVs and that compete with one another. One role for future work might be to explore the effect of market structures other than the one employed in our exploratory model. See section IV.

[10] This is equivalent to the standard assumption in the network literature of a network benefit that is linear in the number of compatible users. See Farrell and Saloner [1992].

11 The constant marginal cost assumption eliminates the price effects of isoelastic increases in demand; this allows the competitive effects on price of the selectivity of externalities to be viewed in isolation.

12 The assumption that the fixed cost, but not marginal cost, of production is affected by λ is analogous to Katz and Shapiro's [1985] assumption that the costs of compatibility are fixed costs (p. 427). Katz and Shapiro address the effects of relaxing the assumption, so I will not do so here.

13 Replacing $C(.)$ with a more general convex form complicates derivation of the equilibrium somewhat, but it does not affect the model's results. Section III covers the effect of using a general cost function for the imposing product versus free product case. A full discussion covering the non-imposing market product case is available from the author on request.

14 One may derive $\bar{v}$ satisfying this requirement (i.e., $\min\{U_A(j)|\bar{v}, U_B(j)|\bar{v}\} > 0$) as follows. As shall be shown, $Q_A + Q_B = 1$ implies v does not appear in the first-order conditions for A's profit maximization. This means $\lambda^*\big|_{Q_A+Q_B=1}$, A's profit-maximizing choice of λ subject to all consumers choosing to purchase A or B, is a function of exogenous parameters other than *v*. Given $p_B = 0$, the expression $\bar{v} \equiv \theta + t + \lambda^*\big|_{Q_A+Q_B=1}$ satisfies.

15 Of course, as is clear from (1) and (2), the *absolute* reservation prices for both A and B decrease with Q_A, in that both become less preferred relative to a choice of "neither A nor B" as Q_A increases. But a network externality with respect to A *relative to B* can be said to exist; and where *v* is large enough to effectively preclude "neither A nor B" as an option, such a network externality has the same market relevance as an "absolute" network externality. Related to this, see also footnote 5 *supra*.

16 Note that our no-tipping condition guarantees $\lambda < 2t/\sigma$, assuring stability in (5). See Appendix.

17 See Appendix.

18 See Appendix for a sketch of the proof.

19 By similar logic to that invoked in footnote 14 above, it is possible to derive a $\bar{v}$ that meets this requirement.

[20] See Appendix.

[21] See Appendix.

[22] One may verify that $\sigma = \frac{1}{2}$ satisfies the no-tipping condition (A8) in the Appendix for the parameter values posited.

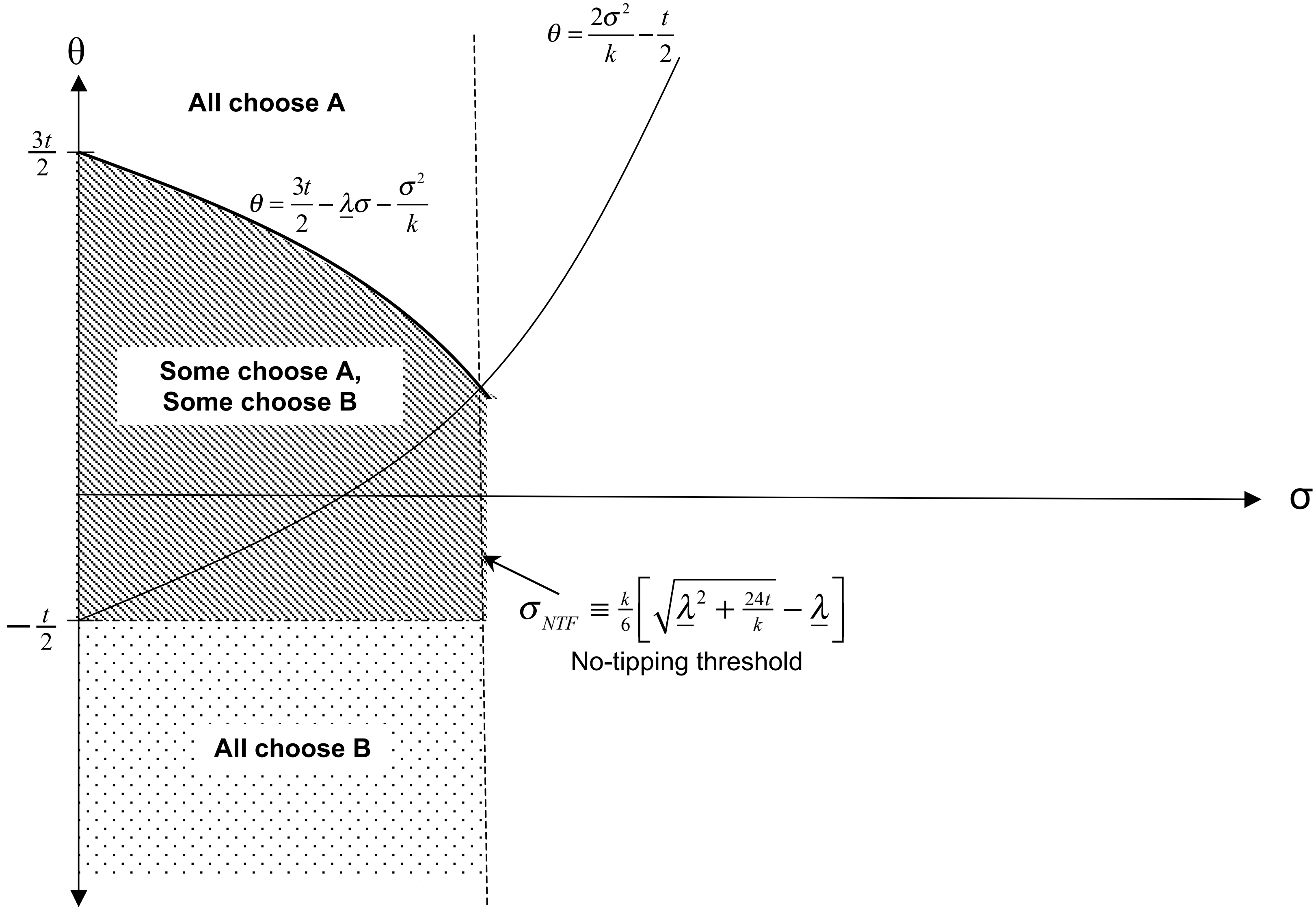


Figure 1
Equilibrium with Externality-Imposing Market Product vs. Free Product

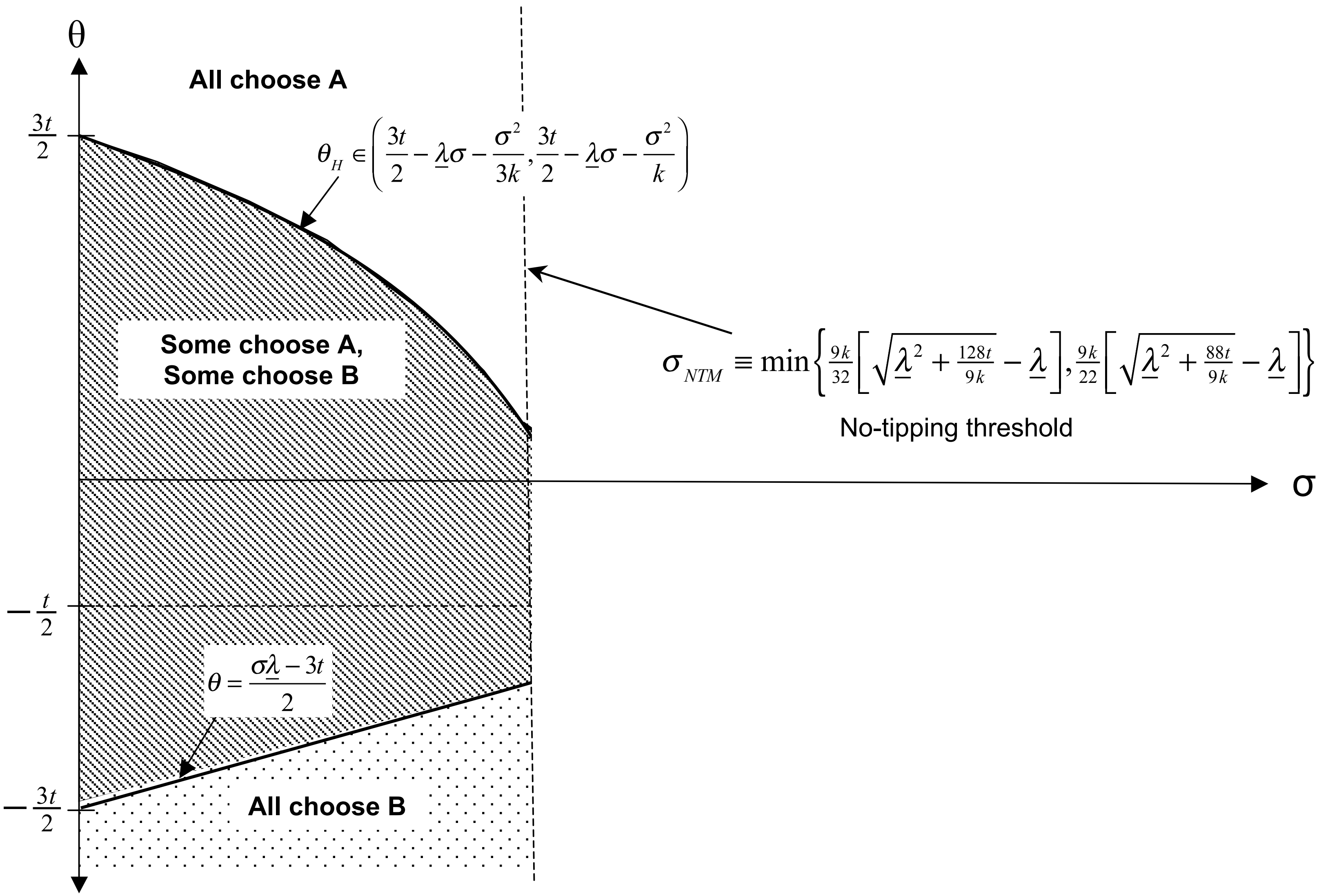


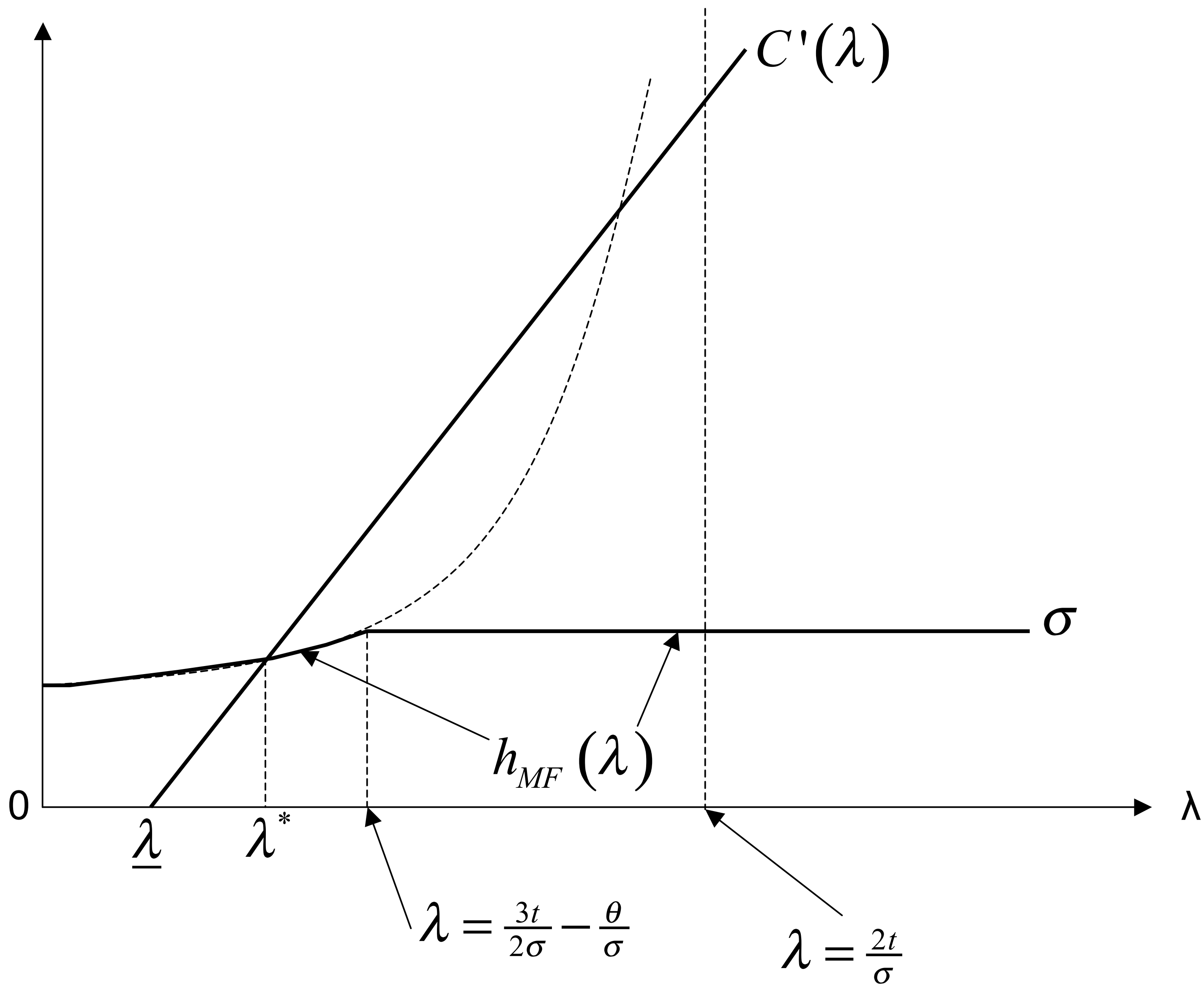
Figure 2
Equilibrium with Externality-Imposing Market Product vs. Non-Imposing Market Product

TABLE I

NUMERICAL EXAMPLE - EQUILIBRIUM WITH EXTERNALITY-IMPOSING MARKET PRODUCT VERSUS NON-IMPOSING MARKET PRODUCT

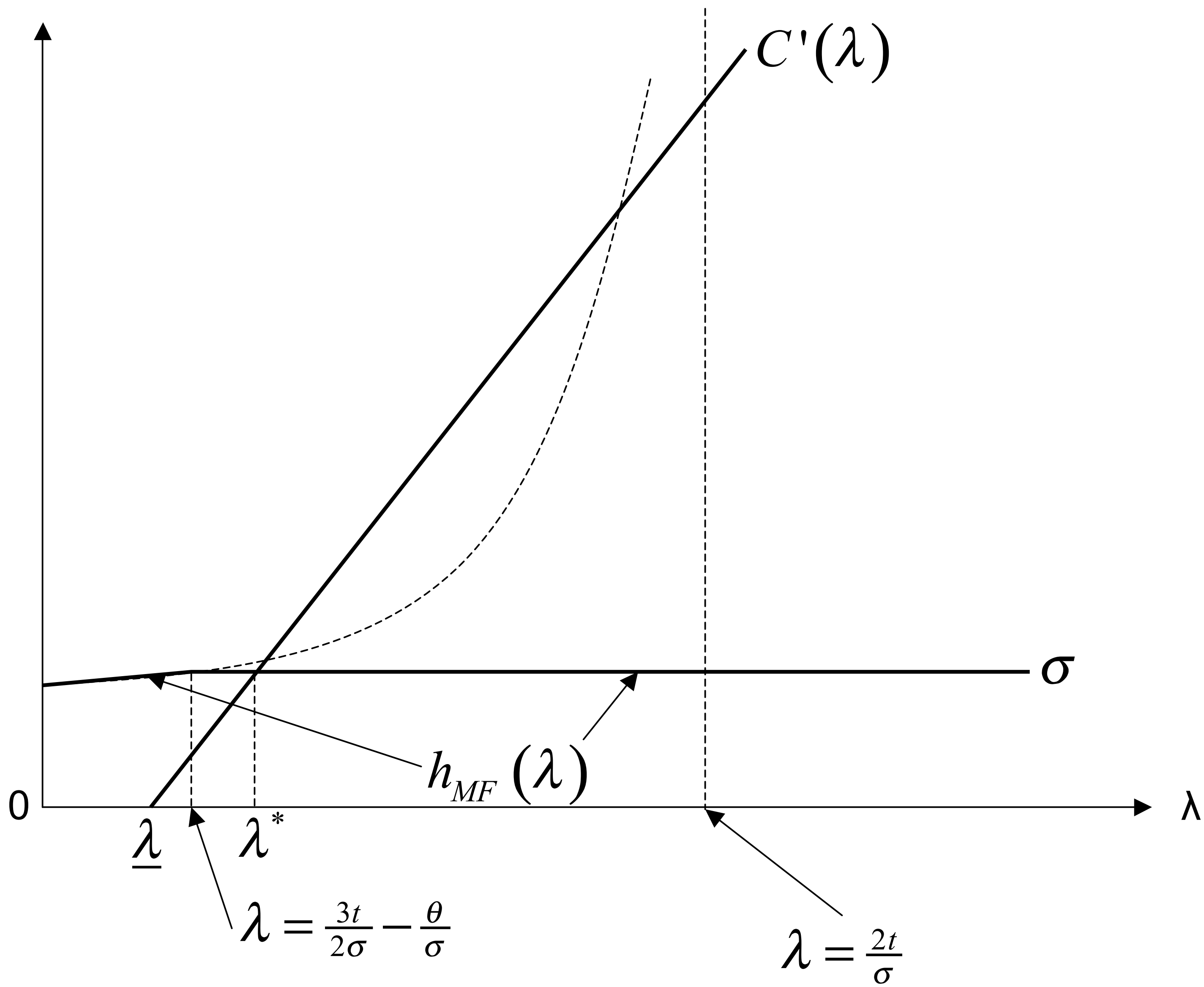
Let $v=7$, $t=1$, $k=1$, $\underline{\lambda}=1$, and $\sigma=\frac{1}{2}$

		At $\lambda=\lambda^*$			At $\lambda=\lambda^*+0.01$			At $\lambda=\lambda^*$	
θ	λ^*	p_A-p_B	Q_A	W	p_A-p_B	Q_A	W	$\frac{\partial Q_A}{\partial \lambda}$ direct effect	$\frac{\partial Q_A}{\partial \lambda}$ indirect effect
0.75	1.111	1.185	0.9102	7.1400	1.187(+)	0.9122(+)	7.1350(-)	0.3150	-0.1154
-0.25	0.950	-0.175	0.4426	6.9466	-0.173(+)	0.4430(+)	6.9432(-)	0.1451	-0.1093
-0.5	0.944	-0.509	0.3333	7.1250	-0.508(+)	0.3333(0)	7.1227(-)	0.1091	-0.1091
-0.75	0.950	-0.842	0.2240	7.3975	-0.840(+)	0.2237(-)	7.3966(-)	0.0735	-0.1093
-1.0	0.969	-1.172	0.1134	7.7696	-1.170(+)	0.1127(-)	7.7703(+)	0.0374	-0.1100
-1.2	0.993	-1.435	0.0230	8.1447	-1.433(+)	0.0220(-)	8.1470(+)	0.0076	-0.1108



$C'(\lambda)$
σ
$h_{MF}(\lambda)$
0
λ
$\underline{\lambda}$
λ^*
$\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$
$\lambda = \frac{2t}{\sigma}$

Figure 3(a)



$C'(\lambda)$
σ
$h_{MF}(\lambda)$
0
λ
$\underline{\lambda}$
λ^*
$\lambda = \frac{3t}{2\sigma} - \frac{\theta}{\sigma}$
$\lambda = \frac{2t}{\sigma}$

Figure 3(b)

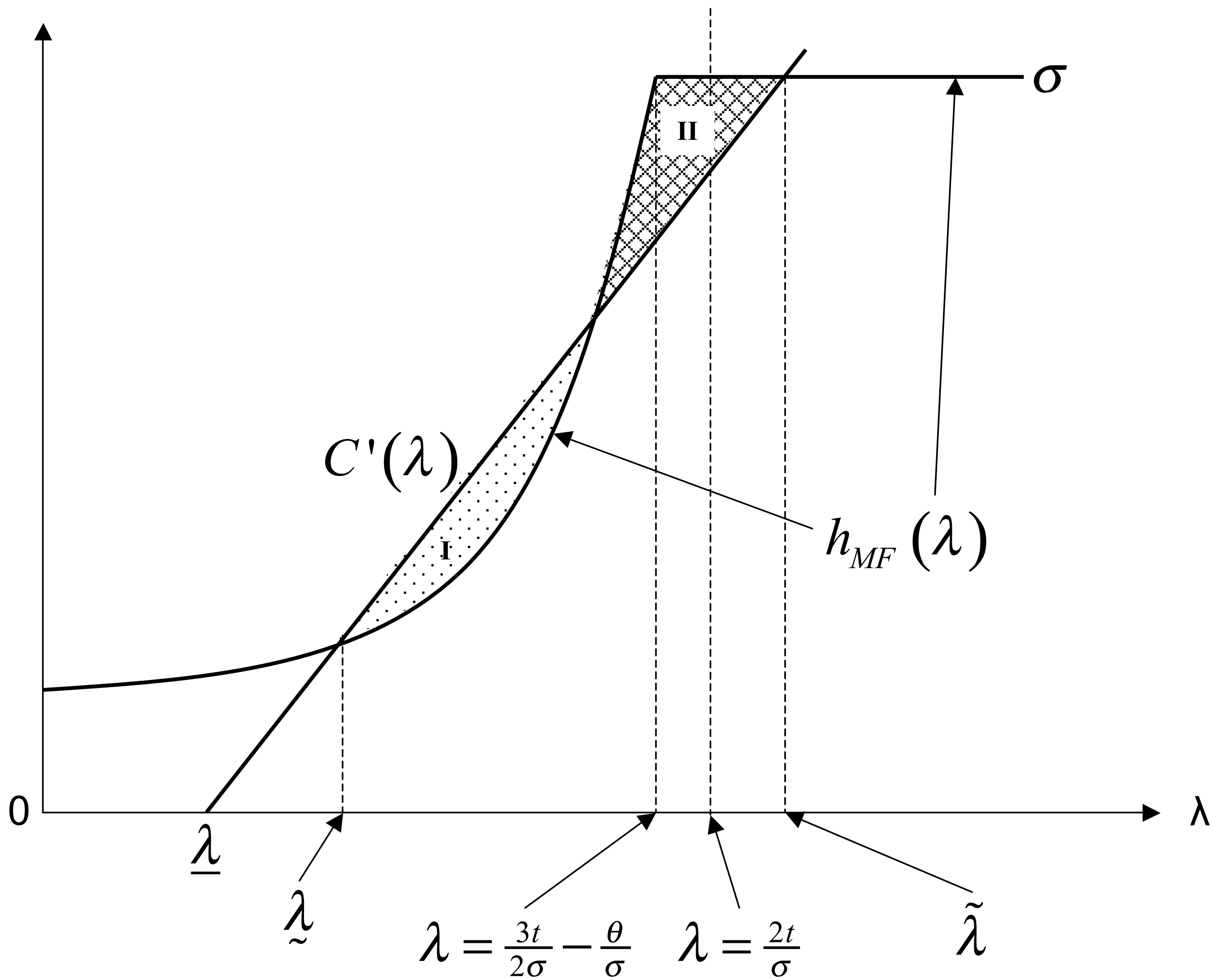
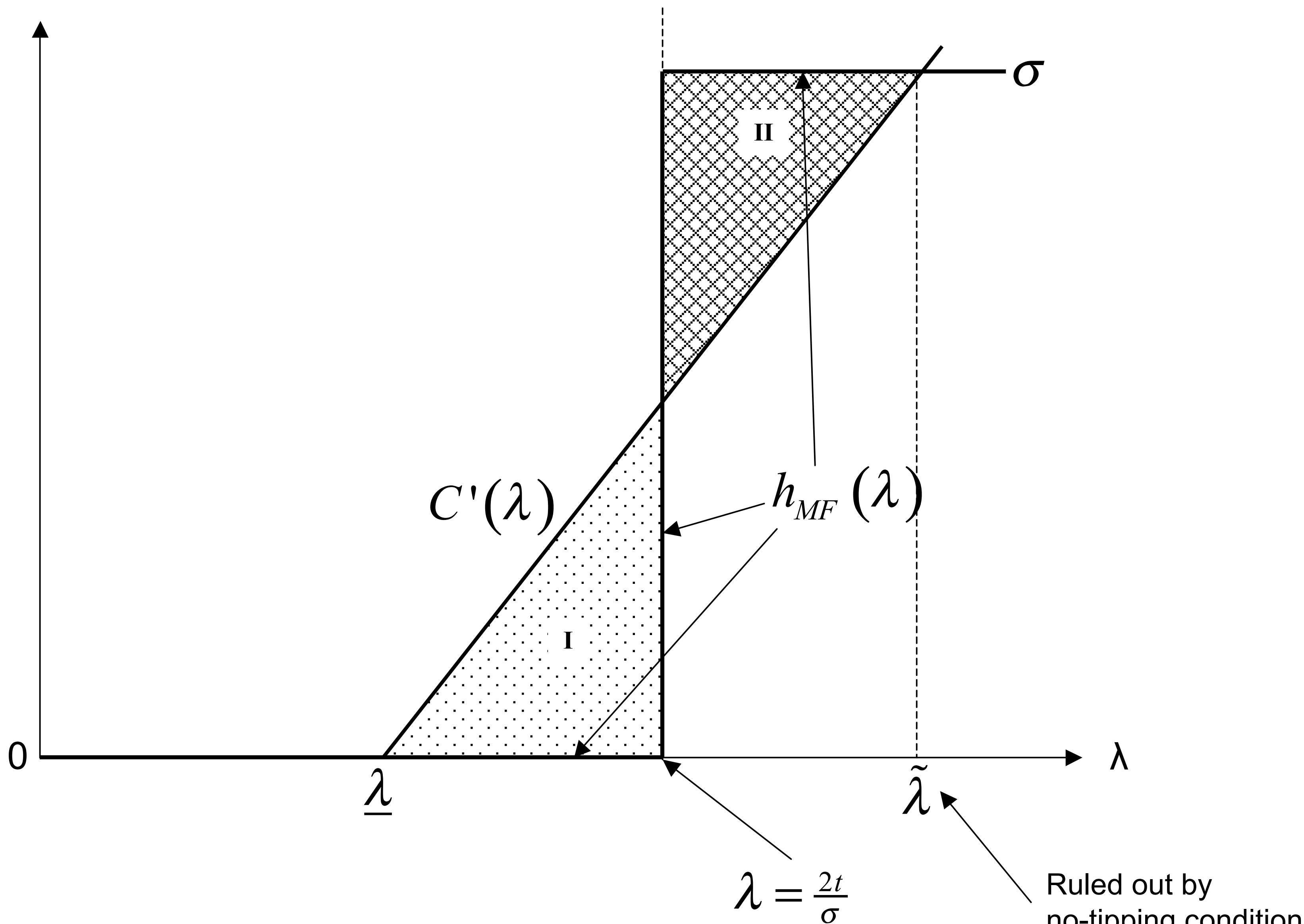


Figure 3(c)
Sub-Case Ruled Out by No-Tipping Condition



σ
II
$h_{MF}(\lambda)$
$C'(\lambda)$
I
0
λ
$\underline{\lambda}$
$\tilde{\lambda}$
$\lambda = \frac{2t}{\sigma}$
Ruled out by
no-tipping condition

Figure 3(d)

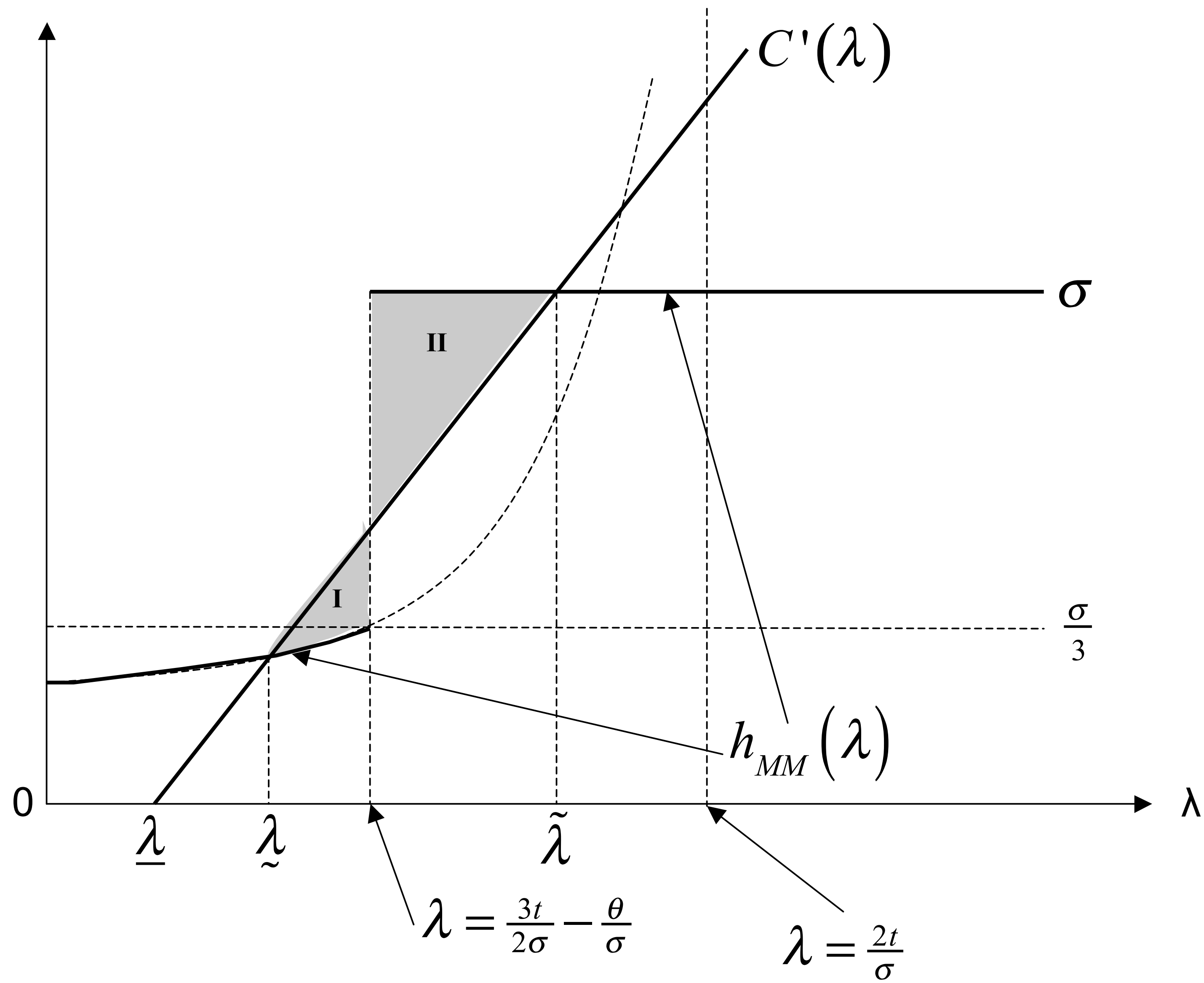
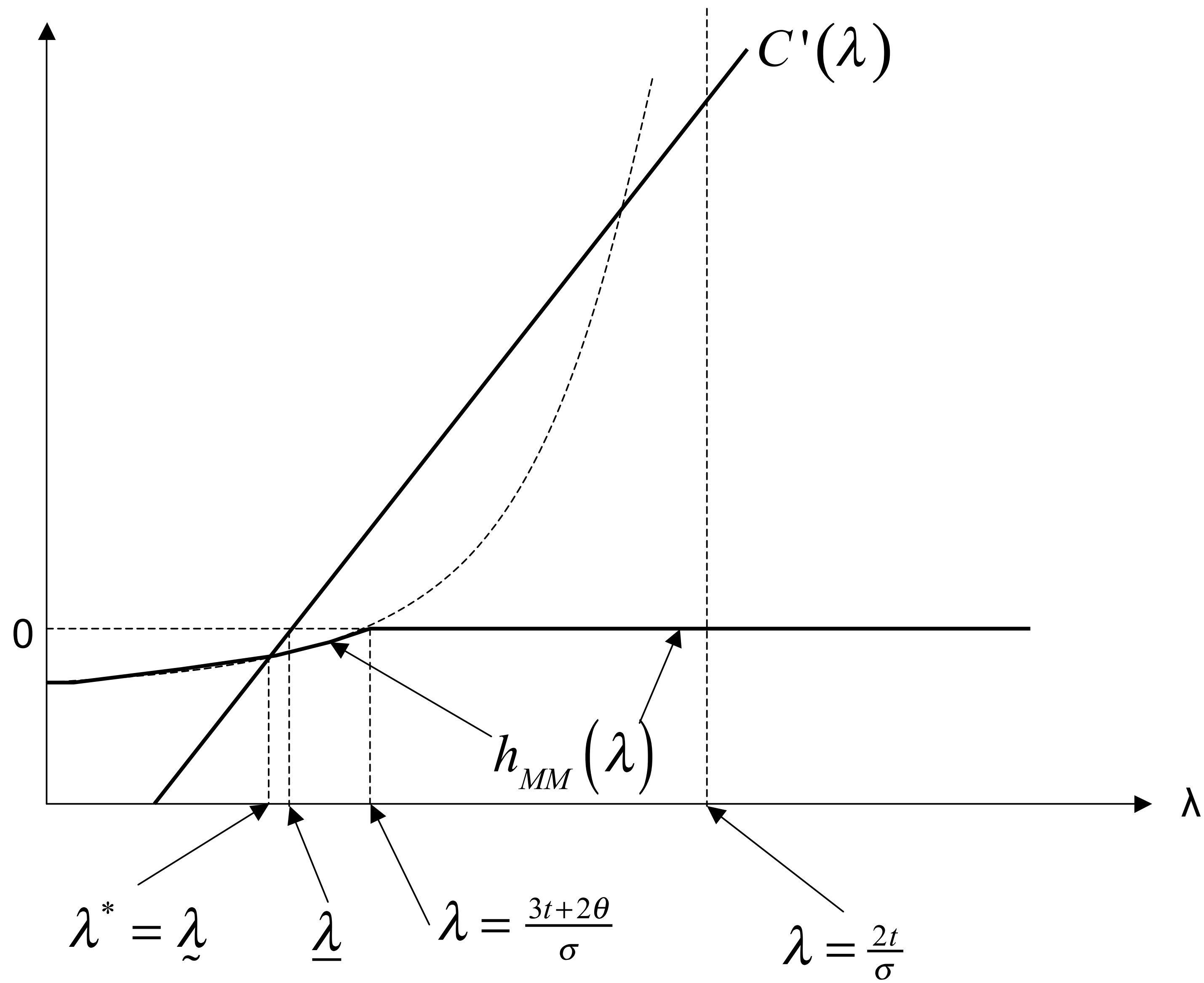
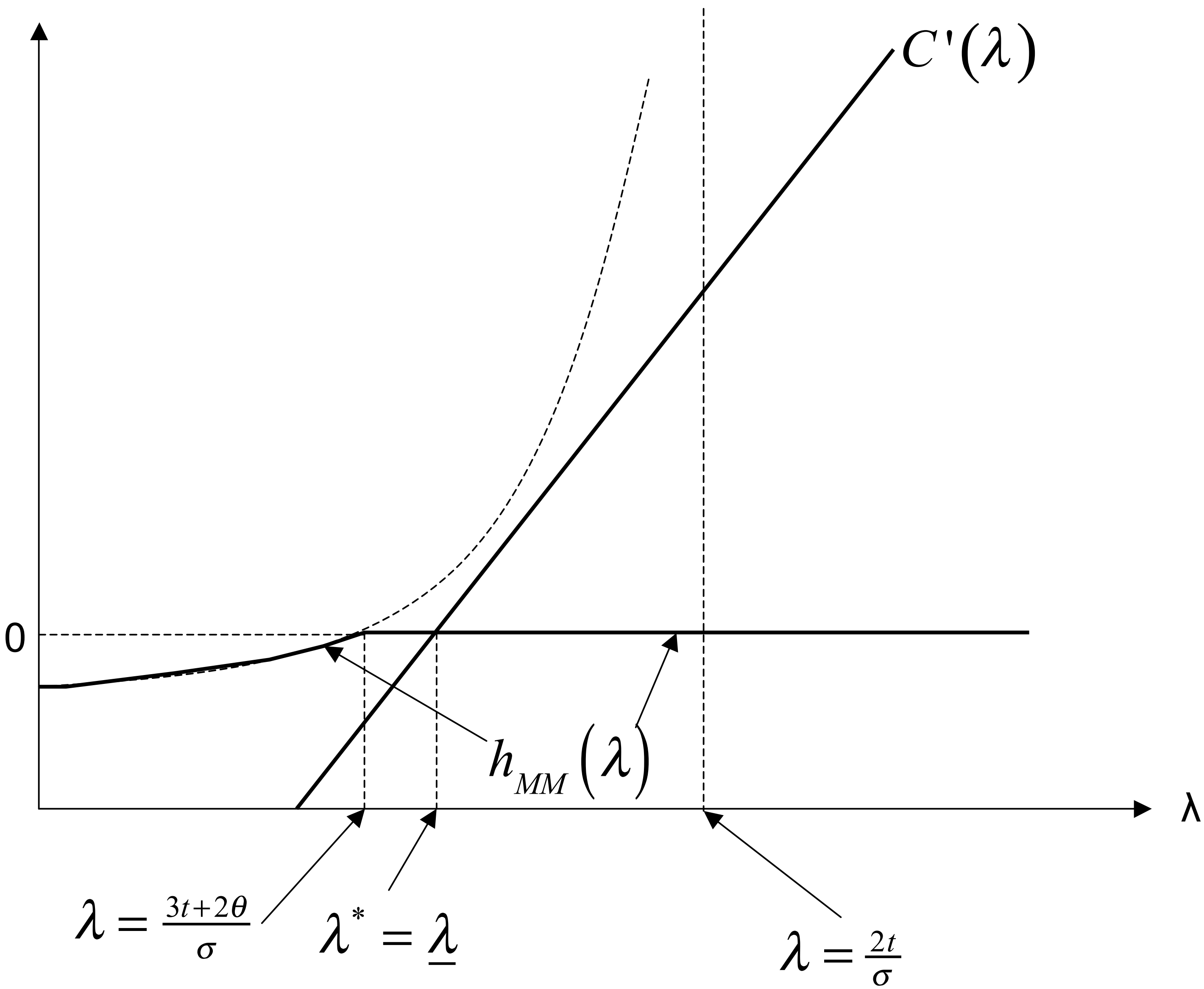


Figure 4



$C'(\lambda)$
0
$h_{MM}(\lambda)$
λ
$\lambda^* = \underset{\sim}{\lambda}$
$\underline{\lambda}$
$\lambda = \frac{3t+2\theta}{\sigma}$
$\lambda = \frac{2t}{\sigma}$

Figure 5(a)



$C'(\lambda)$
0
$h_{MM}(\lambda)$
λ
$\lambda = \frac{3t+2\theta}{\sigma}$
$\lambda^* = \underline{\lambda}$
$\lambda = \frac{2t}{\sigma}$

Figure 5(b)

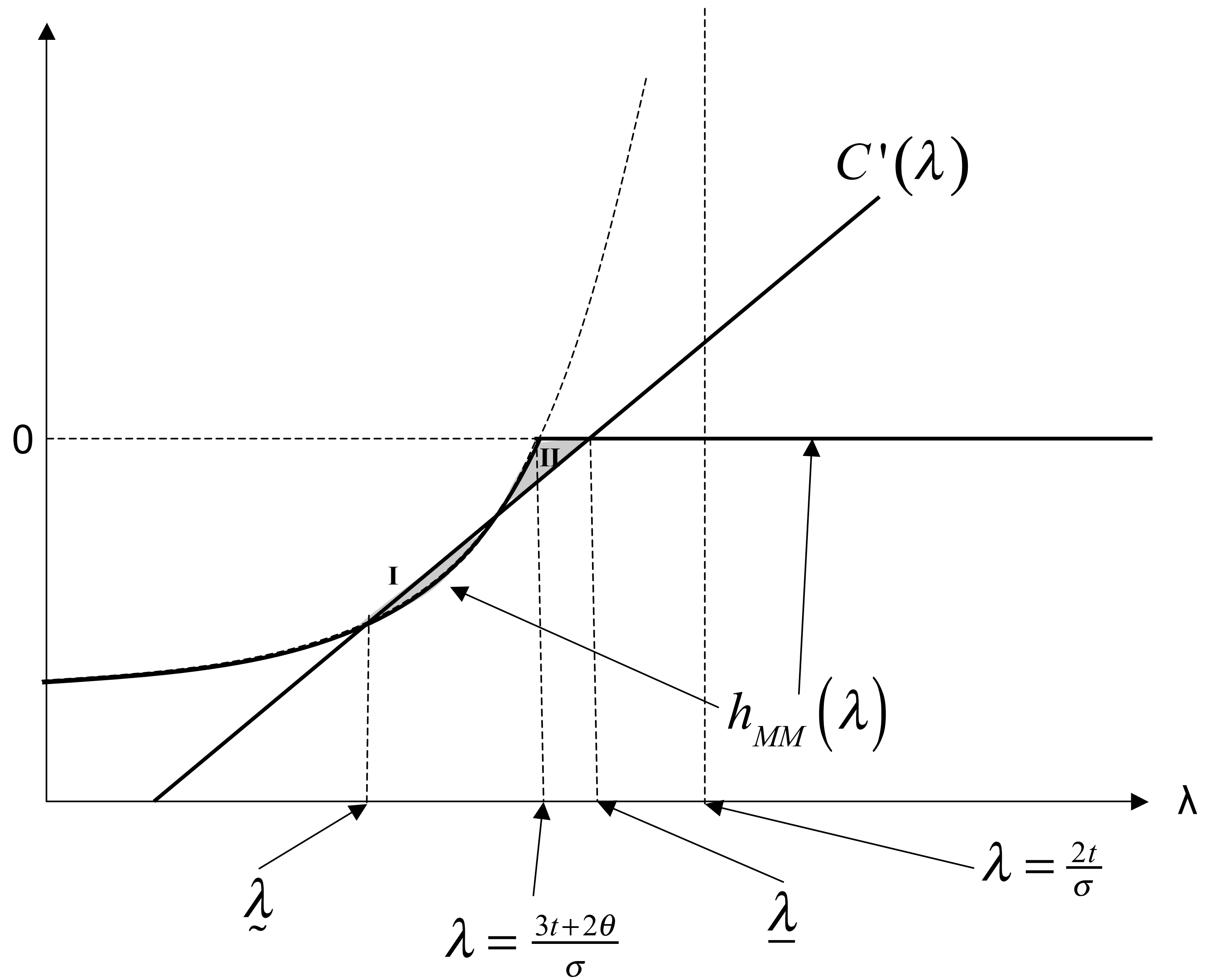


Figure 5(c)
Sub-Case Ruled Out by No-Tipping Condition